RESONANCE WIDTHS FOR THE MOLECULAR PREDISSOCIATION

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Abstract. We consider a semiclassical 2×2 matrix Schrödinger operator of the form $P = -h^2\Delta \mathbf{I}_2 + \mathrm{diag}(V_1(x), V_2(x)) + hR(x, hD_x)$, where V_1, V_2 are real-analytic, V_2 admits a non degenerate minimum at 0, V_1 is non trapping at energy $V_2(0) = 0$, and $R(x, hD_x) = (r_{j,k}(x, hD_x))_{1\leq j,k\leq 2}$ is a symmetric off-diagonal 2×2 matrix of first-order pseudodifferential operators with analytic symbols. We also assume that $V_1(0) > 0$. Then, denoting by e_1 the first eigenvalue of $-\Delta + \langle V_2''(0)x, x\rangle/2$, and under some ellipticity condition on $r_{1,2}$ and additional generic geometric assumptions, we show that the unique resonance ρ_1 of P such that $\rho_1 = V_2(0) + (e_1 + r_{2,2}(0, 0))h + \mathcal{O}(h^2)$ (as $h \to 0_+$) satisfies,

$$\mathrm{Im}\, \rho_1 = -h^{n_0+(1-n_\Gamma)/2} f(h, \ln \frac{1}{h}) e^{-2S/h},$$

where $f(h, \ln \frac{1}{h}) \sim \sum_{0\leq m\leq \ell} f_{\ell,m} h^\ell (\ln \frac{1}{h})^m$ is a symbol with $f_{0,0} > 0$, $S > 0$ is the so-called Agmon distance associated with the degenerate metric $\max(0, \min(V_1, V_2))dx^2$, between 0 and $\{V_1 \leq 0\}$, and $n_0 \geq 1$, $n_\Gamma \geq 0$ are integers that depend on the geometry.

Keywords: Resonances; Born-Oppenheimer approximation; eigenvalue crossing; microlocal analysis.

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1. Introduction

The theory of predissociation goes back to the very first years of quantum mechanics (see, e.g., [Kr, La, Ze, St]). Rouhgly speaking, it describes the possibility for a molecule to dissociate spontaneously (after a sufficiently large time) into several sub-molecules, for energies below the crossing of the corresponding energy surfaces of the initial molecule and the final dissociated state. From a physical point of view, one naturally expects that this (typically quantic) phenomenon occurs with extremely small (but non zero) probability.

Despite the fact that statements concerning this problem are present in the physics literature for more than 70 years, the first mathematically rigorous result is due to M. Klein [Kl] in 1987, where an upper bound on the time of predissociation is given in the framework of the Born-Oppenheimer approximation. More precisely, denoting by h the square root of the ratio of electronic to nuclear mass, M. Klein proves the existence of resonances ρ with real part below the crossing of the energy surfaces, and with exponentially small imaginary part, that is,

$$|\mathrm{Im}\ \rho| = \mathcal{O}(e^{-2(1-\varepsilon)S/h})$$

where $S > 0$ is a geometric constant, $\varepsilon > 0$ is fixed arbitrarily, and the estimate holds uniformly as h goes to zero.

In terms of probabilities, this result corresponds to give an upper bound on the transition probability between the initial molecule and the dissociated state. The purpose of this article is to obtain a more complete information on this quantity, and in particular a lower bound on it. More precisely, under suitable conditions, we prove that the imaginary part of the lowest resonance admits a complete asymptotic expansion of the type,

$$\mathrm{Im}\ \rho_1 = -h^{n_0+(1-n_\Gamma)/2}e^{-2S/h}\sum_{0\leq m\leq \ell} f_{\ell,m}h^\ell(\ln\frac{1}{h})^m,$$

in the sense that, for any $N \geq 1$, one has

$$|\mathrm{Im}\ \rho_1 + h^{n_0+(1-n_\Gamma)/2}e^{-2S/h}\sum_{0\leq m\leq \ell\leq N} f_{\ell,m}h^\ell(\ln\frac{1}{h})^m| = \mathcal{O}(h^{n_0+(1-n_\Gamma)/2+N}e^{-2S/h}),$$

where $S > 0$, $n_0 \geq 1$ and $n_\Gamma \geq 0$ are all geometric constants, and where the leading coefficient $f_{0,0}$ is positive.

As it is well-known, the quantity $\mathrm{Im}\,\rho$ is closely related to the oscillatory behavior of the corresponding resonant state in the unbounded classically allowed region. Hence, the main issue will be to know sufficiently well this behavior.

The strategy of the proof consists in starting from the WKB construction at the bottom of the well, and then trying to extend them as much as possible, and at least up to the classically allowed unbounded region. This is mainly the same strategy used in [HeSj2] for the study of shape resonances.

However, from a technical point of view, several new problems are encountered, because of the crossing of the electronic levels.

The first one is that, at the crossing, the only reference on WKB constructions is that of [Pe], that has been done for a special type of matrix Schrödinger operators. In particular, it strongly uses the fact that only differential operators are involved. In our case, since our operator comes from a Born-Oppenheimer reduction, it is necessarily of pseudodifferential kind (see, e.g., [KMSW, MaSo]). As a consequence, our first step will consist in extending the method of [Pe] to pseudodifferential operators. Unfortunately, this extension is far from being straightforward, and needs a specific formal calculus adapted to expressions involving the Weber functions.

The second one is that, after having overcome the crossing, the symbols of the resulting WKB expansions do not anymore satisfy analytic estimates (usually needed in order to re-sum them, up to exponentially small error terms). In particular, this prevents us from using directly the constructions of [HeSj2] near the classically allowed unbounded region. Instead, we have to adapt the method of [FLM] that, without analyticity, allows us to extend the WKB constructions into the classically allowed unbounded region up to a distance of order $(h\ln|h|)^{2/3}$ from the barrier. This is not much, but this is enough for having a sufficient control, in this region, on the difference between the true solution and the WKB one. This is actually done by adapting the specific arguments of propagation introduced in [FLM], where the propagation takes place in h-dependent domains.

In the next section, we describe in details the geometrical context and the assumptions.

In Section 3, we state our main result.
Section 4 is devoted to the WKB constructions, starting from the well and proceeding away along some minimal geodesics, until crossing the boundary of the classically forbidden region. It is in this section that we develop a formal pseudodifferential calculus adapted to expressions involving the Weber functions.
Next, in Section 5, we extend the well-known Agmon estimates to our pseudodifferential context. In this case, the main feature is that, since we cannot use general Lipschitz weight-functions, we replace them by h-dependent smooth functions with bounded gradient, but with derivatives of higher order that can grow to infinity as $h \to 0$.
In Section 6, we use these estimates in order to obtain a bound for the difference between the WKB solutions and a solution of a modified problem, and this permits us to define an asymptotic solution in a whole neighborhood of the classically forbidden region (but only up to a distance of order $(h \ln |h|)^{1/3}$ from this region).
Section 7 contains the *a priori* estimates and the propagation arguments that lead to a good control on the difference between the asymptotic solution and the actual one.
Finally, Section 8 makes the link with the width of the resonance. Even if the idea is standard (practically an application of the Green formula: see, e.g., [HeSj2]), here we have to be careful with the double problem that, on the one hand, we deal with pseudodifferential (not differential) operators, and, on the other hand, the magnitude of freedom outside the classically forbidden region is of order $(h \ln |h|)^{1/3}$ as $h \to 0$.

2. Geometrical Assumptions

We consider the semiclassical 2×2 matrix Schrödinger operator,

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} + hR(x, hD_x) \tag{2.1}$$

with,

$$P_j := -h^2\Delta + V_j(x) \quad (j = 1, 2),$$

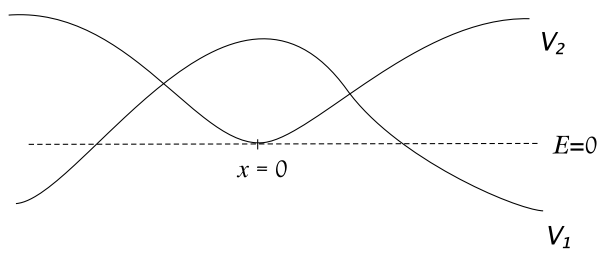
where $x = (x_1, \dots, x_n)$ is the current variable in $\mathbb{R}^n$ $(n \geq 1)$, $h > 0$ denotes the semiclassical parameter, and $R(x, hD_x) = (r_{j,k}(x, hD_x))_{1 \leq j,k \leq 2}$ is a formally self-adjoint 2×2 matrix of first-order semiclassical pseudodifferential operators.
Let us observe that this is typically the kind of operator one obtains in the Born-Oppenheimer approximation, after reduction to an effective Hamiltonian (see [KMSW, MaSo]). In that case, the quantity h^2 stands for the inverse of the mass of the nuclei.

Assumption 1. *The potentials V_1 and V_2 are smooth and bounded on $\mathbb{R}^n$, and satisfy,*

$$V_1(0) > 0 \text{ and } E = 0 \text{ is a non-trapping energy for } V_1; \tag{2.2}$$

$$V_1 \text{ has a strictly negative limit as } |x| \to \infty; \tag{2.3}$$

$$V_2 \geq 0 \; ; \; V_2^{-1}(0) = \{0\} \; ; \; \mathrm{Hess} V_2(0) > 0 \; ; \; \liminf_{|x| \to \infty} V_2 > 0. \tag{2.4}$$



In particular we assume that V_2 has a unique non degenerate well at $x = 0$. This well is included in the island $\ddot{O}$ which is the bounded open set defined as,

$$\ddot{O} = \{x \in \mathbb{R}^n; V_1(x) > 0\}. \tag{2.5}$$

Next, we define the sea as the set where $V_1(x) < 0$.
The fact that 0 is a non-trapping energy for V_1 means that, for any $(x, \xi) \in p_1^{-1}(0)$, one has $|\exp tH_{p_1}(x, \xi)| \to +\infty$ as $t \to \infty$, where $p_1(x, \xi) := \xi^2 +$

$V_1(x)$ is the symbol of P_1, and $H_{p_1} := (\nabla_\xi p_1, -\nabla_x p_1)$ is the Hamilton field of p_1.

Such conditions (2.2)-(2.4) correspond to the situation of *molecular predissociation* as described in [Kl].

Since we plan to study the resonances of P near the energy level $E = 0$, we also assume,

Assumption 2. *The potentials V_1 and V_2 extend to bounded holomorphic functions near a complex sector of the form, $\mathcal{S}_{R_0,\delta} := \{x \in \mathbb{C}^n\,;\, |\mathrm{Re}\, x| \geq R_0\,,\, |\mathrm{Im}\, x| \leq \delta |\mathrm{Re}\, x|\}$, with $R_0, \delta > 0$. Moreover V_1 tends to its limit at ∞ in this sector and* Re V_2 *stays away from* 0 *in this sector.*

Assumption 3. *The symbols $r_{j,k}(x,\xi)$ for $(j,k) = (1,1), (1,2), (2,2)$ extend to holomorphic functions near,*

$$\widetilde{\mathcal{S}}_{R_0,\delta} := \mathcal{S}_{R_0,\delta} \times \{\xi \in \mathbb{C}^n\,;\, |\mathrm{Im}\, \xi| \leq \max(\delta\langle \mathrm{Re}\, x\rangle, \sqrt{M_0})\},$$

with,

$$M_0 > \sup_{x\in\mathbb{R}^n} \min(V_1(x), V_2(x)).$$

and, for any $\alpha \in \mathbb{N}^{2n}$, they satisfy

$$\partial^\alpha r_{j,k}(x,\xi) = \mathcal{O}(\langle \mathrm{Re}\, \xi\rangle) \quad \textit{uniformly on } \widetilde{\mathcal{S}}_{R_0,\delta}. \tag{2.6}$$

Now we define the cirque Ω as,

$$\Omega = \{x \in \mathbb{R}^n ; V_2(x) < V_1(x)\} \tag{2.7}$$

Hence, the well is in the cirque and the cirque is in the island.

We also consider the Agmon distance associated to the pseudo-metric

$$(\min(V_1, V_2))_+ dx^2.$$

Such a metric is considered in Pettersson [Pe]. There are three places where this metric is not a standard one.

First, near the well 0, but this case is well known. It has been treated by Helffer and Sjöstrand [HeSj1]. It is also considered in Pettersson. The Agmon distance,

$$\varphi(x) = d(x, 0) \tag{2.8}$$

is smooth at 0. The point $(x,\xi) = (0,0)$ is a hyperbolic singular point of the Hamilton vector field H_{q_2}, where $q_2 = \xi^2 - V_2(x)$, and the stable and

unstable manifold near this point are respectively the Lagrangian manifolds $\{\xi = \nabla\varphi(x)\}$ and $\{\xi = -\nabla\varphi(x)\}$.
Secondly on $\partial\Omega$, precisely at the points where $V_1 = V_2$. This case has been also considered in Pettersson. At such a point, if one assume that $\nabla V_1 \neq \nabla V_2$, then, any geodesic which is tranversal to the hypersurface $\{V_1 = V_2\}$ is C^1.
Finally there is the boundary of the island $\partial\ddot{O}$, where $V_1 = 0$. This situation was considered in Helffer-Sjöstrand [HeSj2]. We will follow them in a next assumption.
Now we consider the distance from the well to the sea, that means to $\partial\ddot{O}$:

$$S = d(0, \partial\ddot{O}), \tag{2.9}$$

Setting $B_S = \{x \in \ddot{O}, \varphi(x) < S\}$ and denoting by $\bar{B}_S$ its closure, we also consider the set $\bar{B}_S \cap \partial\ddot{O}$, that consists in the points of the boundary of the island that are joined to the well by a minimal d-geodesic included in the island. These points are called points of type 1 in [HeSj2], and we denote by G the set of the minimal geodesics joining such a point to 0 in $\ddot{O}$.

We make the following assumption.

Assumption 4. *For all $\gamma \in G$, γ intersects $\partial\Omega$ at a finite number of points and the intersection is transversal at each of these points. Moreover, $\nabla V_1 \neq \nabla V_2$ on $\gamma \cap \partial\Omega$.*

Let us recall that the assumption that 0 is a non trapping energy for V_1 implies that $\nabla V_1 \neq 0$ on $\partial\ddot{O}$, and therefore that $\partial\ddot{O}$ is a smooth hypersurface.

We define the caustic set $\mathcal{C}$ as the closure of the set of points $x \in \ddot{O}$ with $\varphi(x) = S + d(x, \partial\ddot{O})$. In particular, the points of type 1 are in $\mathcal{C}$. As in [HeSj2] we assume,

Assumption 5. *The points of type 1 form a submanifold Γ, and $\mathcal{C}$ has a contact of order exactly two with $\partial\ddot{O}$ along Γ.*

We denote by n_Γ the dimension of Γ. Moreover, for any $\gamma \in G$, we denote by $N_\gamma := \#(\gamma \cap \partial\Omega)$ the number of points where γ crosses the boundary of the cirque, and we set,

$$n_0 := \min_{\gamma \in G} N_\gamma \quad ; \quad G_0 := \{\gamma \in G \,;\, N_\gamma = n_0\}.$$

Then, we make an assumption that somehow insures that an interaction between the two Schrödinger operators does exist.

Assumption 6. *There exists at least one* $\gamma \in G_0$ *for which one has the ellipticity condition* $r_{12}(x, i\nabla\varphi(x)) \neq 0$ *at every point* $x \in \gamma \cap \partial\Omega$.

3. Main Result

Under the previous assumption we plan to study the resonances of the operator P given in (2.1), where $R(x, hD_x)$ is defined as

$$R(x, hD_x) := \begin{pmatrix} \mathrm{Op}_h^L(r_{1,1}) & \mathrm{Op}_h^L(r_{1,2}) \\ \mathrm{Op}_h^R(\overline{r_{1,2}}) & \mathrm{Op}_h^L(r_{2,2}) \end{pmatrix}$$

where for any symbol $a(x, \xi)$ we use the following quantizations,

$$\mathrm{Op}_h^L(a)u(x) = \frac{1}{(2\pi h)^n} \int e^{i(x-y)\xi/h} a(x, \xi) u(y) dy d\xi;$$
$$\mathrm{Op}_h^R(a)u(x) = \frac{1}{(2\pi h)^n} \int e^{i(x-y)\xi/h} a(y, \xi) u(y) dy d\xi.$$

In order to define the resonances we consider the distortion given as follows: Let $F(x) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $F(x) = 0$ for $|x| \leq R_0$, $F(x) = x$ for $|x|$ large enough. For $\theta > 0$ small enough, we define the distorded operator P_θ as the value at $\nu = i\theta$ of the extension to the complex of the operator $U_\nu P U_\nu^{-1}$ which is defined for ν real small enough, and analytic in ν, where we have set

$$U_\nu \phi(x) = \det(1 + \nu dF(x))^{1/2} \phi(x + \nu F(x)). \tag{3.1}$$

Since we have a pseudodifferential operator $R(x, hD)$ the fact that $U_\nu P U_\nu^{-1}$ is analytic in ν is not completely standard but can be done without problem (thanks to Assumption 3), and by using the Weyl Perturbation Theorem, one can also see that there exists $\varepsilon_0 > 0$ such that for any $\theta > 0$ small enough, the spectrum of P_θ is discrete in $[-\varepsilon_0, \varepsilon_0] - i[0, \varepsilon_0\theta]$. The eigenvalues of P_θ are called the resonances of P [Hu, HeSj2, HeMa].

We will need another small parameter $k > 0$ related to the semiclassical parameter $h > 0$, defined as,

$$k := h \ln \frac{1}{h}. \tag{3.2}$$

In the sequel, we will study the resonances in the domain $[-\varepsilon_0, Ch] - i[0, Ck]$, where $C > 0$ is arbitrarily large. In this case, we can adapt the WKB constructions near the well made in [HeSj1], and show that these resonances

form a finite set $\{\rho_1, \dots, \rho_m\}$, with asymptotic expansions as $h \to 0$, of the form,

$$\rho_j \sim h \sum_{\ell \geq 0} \rho_{j,\ell} h^{\ell/2},$$

where $\rho_{j,\ell} \in \mathbb{R}$ and $\rho_{j,0} = e_j + r_{2,2}(0,0)$, e_j being the j-th eigenvalue of the harmonic oscillator $-\Delta + \langle V_2''(0)x, x\rangle/2$ (actually, to be more precise, one must also assume that the arbitrarily large constant C does not coincide with one of the e_j's).

In this paper we are interested in the imaginary part of these resonances. We have,

Theorem 3.1. *Under Assumptions 1 to 6, the first resonance ρ_1 of P is such that,*

$$\mathrm{Im}\ \rho_1 = -h^{n_0 + (1-n_\Gamma)/2} f(h, \ln \frac{1}{h}) e^{-2S/h}$$

where $f(h, \ln \frac{1}{h})$ admits an asymptotic expansion of the form,

$$f(h, \ln \frac{1}{h}) \sim \sum_{0 \leq m \leq \ell} f_{\ell,m} h^{\ell} (\ln \frac{1}{h})^m, \quad (h \to 0),$$

with $f_{0,0} > 0$, and $S > 0$ is defined in (2.9).

Moreover the other resonances in $[-\varepsilon_0, Ch] - i[0, Ck]$ verify

$$\mathrm{Im}\ \rho_j = \mathcal{O}(h^{\beta_j} e^{-2S/h}),$$

for some real β_j, uniformly as $h \to 0$.

4. WKB constructions

In this chapter, we fix some minimal d-geodesic $\gamma \in G$, and we denote by $x^{(1)}, \dots, x^{(N_\gamma)}$ the sequence of points that constitute $\gamma \cap \partial\Omega$, ordered from the closest to 0 up to the closest to $\ddot{O}$ (note that N_γ is necessarily an odd number). We also denote by $\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(N_\gamma+1)}$ the portions of $\gamma \backslash \partial\Omega$ that are in-between 0 and $x^{(1)}$, $x^{(1)}$ and $x^{(2)}$, ..., $x^{(N_\gamma)}$ and $\ddot{O}$, respectively, in such a way that we have,

$$\gamma = \gamma^{(1)} \cup \{x^{(1)}\} \cup \gamma^{(2)} \cup \dots \cup \{x^{(N_\gamma)}\} \cup \gamma^{(N_\gamma+1)},$$

where the union is disjoint (in particular, by convention we assume that $0 \in \gamma^{(1)}$). Moreover, we start by considering the first resonance ρ_1 only.

4.1. **In the cirque.** As in [Pe], the starting point of the construction consists in the WKB asymptotics given near the well $x = 0$ by a method due to Helffer and Sjöstrand [HeSj1]. More precisely, because of the matricial nature of the operator and the fact that p_1 is elliptic above $x = 0$, one finds a formal solution w_1 of $Pw_1 = \rho_1 w_1$ of the form,

$$w_1(x;h) = \begin{pmatrix} ha_1(x,h) \\ a_2(x,h) \end{pmatrix} e^{-\varphi(x)/h}. \tag{4.1}$$

where φ is defined in (2.8), and a_j $(j = 1, 2)$ is a classical symbol of order 0 in h, that is, a formal series in h of the form,

$$a_j(x,h) = \sum_{k=0}^{\infty} h^k a_{j,k}(x) \tag{4.2}$$

with $a_{j,k}$ smooth near 0 (here no half-powers of h appear since we consider the first resonance ρ_1 only). Moreover, a_2 is elliptic in the sense that $a_{2,0}$ never vanishes. Note that the generalization of the constructions of [HeSj1] to the case of pseudodifferential operators is done by the use of a so-called *formal semiclassical pseudodifferential calculus*, which in our case is based on the following result:

Lemma 4.1. *Let $\widetilde{\varphi} = \widetilde{\varphi}(x)$ be a real bounded C^∞ function on $\mathbb{R}^n$ and let $p = p(x,\xi) \in S(1)$, that extends to a bounded function, holomorphic with respect to ξ in a neighborhood of the set,*

$$\{(x,\xi) \in \mathrm{Supp}\nabla\widetilde{\varphi} \times \mathbb{C}^n \; ; \; |\mathrm{Im}\,\xi| \leq |\widetilde{\nabla}\varphi(x)|\}.$$

Then, denoting by Op_h^L the left (or standard) semiclassical quantization of symbols, the operator $e^{\widetilde{\varphi}/h}\mathrm{Op}_h^L(p)e^{-\widetilde{\varphi}/h}$ is uniformly bounded on $L^2(\mathbb{R}^n)$, and for any $a \in C_0^\infty(\mathbb{R}^n)$ and $N \geq 1$, one has,

$$\begin{aligned} &\left(e^{\widetilde{\varphi}/h}\mathrm{Op}_h^L(p)e^{-\widetilde{\varphi}/h}a\right)(x;h) \\ &\quad = \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\left(\frac{h}{i}\right)^{|\alpha|} \partial_\xi^\alpha p(x, i\nabla\widetilde{\varphi}(x))\partial_y^\alpha \left(a(y)e^{\Phi(x,y)/h}\right)_{y=x} + \mathcal{O}(h^{N/2}), \end{aligned} \tag{4.3}$$

locally uniformly with respect to x, and uniformly with respect to h small enough. Here, $\Phi(x,y) := \widetilde{\varphi}(x) - \widetilde{\varphi}(y) - (x-y)\widetilde{\nabla}\varphi(x)$.

The proof of this lemma is rather standard and we omit it (see e.g. [Ma1]). Then, the construction can be performed by using the formal series given in (4.3) in order to define the formal action of $R(x, hD_x)$ on w_1. Afterwards, these constructions can be continued along the integral curves of the vector

field $\nabla_\xi p_2(x, i\nabla\varphi(x))D_x = 2\nabla\varphi(x).\nabla_x$ (that is, along the minimal geodesic of d starting at 0), as long as $p_1(x, i\nabla\varphi(x))$ does not vanish (that is, as long as these minimal geodesics stay inside the cirque Ω). In that way, after resummation and multiplication by a cut-off function, we obtain a function w_1 of the form (4.1), that satisfies,

$$Pw_1 - \rho_1 w_1 = \mathcal{O}(h^\infty e^{-\varphi/h}), \tag{4.4}$$

locally uniformly in $\bigcup \gamma$, where the union is taken over all the minimal d-geodesics γ coming from the well 0 and staying in Ω. In particular, (4.4) is satisfied in a neighborhood $\mathcal{N}_1$ of $\gamma^{(1)}$.

4.2. **At the boundary of the cirque.** Now, we study the situation near the point $x^{(1)} \in \partial\Omega$. By [Pe], Theorem 2.14, we know that there exists a neighborhood $\mathcal{V}_1$ of $x^{(1)}$ and two positive functions $\varphi_1, \varphi_2 \in C^\infty(\mathcal{V}_1)$ such that,

$$\begin{aligned}
&\varphi_1 = \varphi \text{ on } \mathcal{V}_1 \cap \{V_1 < V_2\};\\
&\varphi_2 = \varphi \text{ on } \mathcal{V}_1 \cap \{V_2 < V_1\};\\
&|\nabla\varphi_j(x)|^2 = V_j(x) \quad (j = 1, 2);\\
&\varphi_1 = \varphi_2 \text{ and } \nabla\varphi_1 = \nabla\varphi_2 \text{ on } \mathcal{V}_1 \cap \partial\Omega;\\
&\varphi_2(x) - \varphi_1(x) \sim d(x, \partial\Omega)^2.
\end{aligned}$$

Actually, φ_2 is nothing but $d_2(0, x)$, where d_2 is the Agmon distance associated with the metric $V_2(x)dx^2$, and φ_1 is the phase function of the Lagrangian manifold obtained as the flux-out of $\{(x, \nabla\varphi_2(x))\,;\, x \in \mathcal{V}_1 \cap \partial\Omega\}$ under the Hamilton flow of $q_1(x, \xi) := \xi^2 - V_1(x)$.

Then, we set,

$$\psi := \frac{1}{2}(\varphi_1 + \varphi_2), \tag{4.5}$$

and we consider the smooth function $z(x)$ defined for $x \in \mathcal{V}_1$ by,

$$\begin{aligned}
&z(x)^2 = 2(\varphi_2(x) - \varphi_1(x))\\
&z(x) < 0 \text{ on } \mathcal{V}_1 \cap \{V_2 < V_1\}.
\end{aligned} \tag{4.6}$$

In order to extend the WKB construction (4.1) accross $\partial\Omega$ near $x^{(1)}$, we still follow [Pe] and try a formal ansatz of the form,

(4.7)
$$w_2(x;h) = \sum_{k\geq 0} h^k \left(\alpha_k(x,h)Y_{k,0}\left(\frac{z(x)}{\sqrt{h}}\right) + \sqrt{h}\beta_k(x,h)Y_{k,1}\left(\frac{z(x)}{\sqrt{h}}\right)\right) e^{-\psi(x)/h},$$

where,

$$\alpha_k(x,h) = \begin{pmatrix} h\alpha_{k,1}(x,h) \\ \alpha_{k,2}(x,h) \end{pmatrix} \quad ; \quad \beta_k(x,h) = \begin{pmatrix} \beta_{k,1}(x,h) \\ h\beta_{k,2}(x,h) \end{pmatrix}, \tag{4.8}$$

$\alpha_{k,j}$ and $\beta_{k,j}$ are formal symbols of the form,

$$\sum_{l\geq 0}\sum_{m=0}^{l} h^l(\ln h)^m \gamma^{l,m}(x) \tag{4.9}$$

(with $\gamma^{l,m}$ smooth in ω_1), and for any $k \geq 0$ and $\varepsilon \in \mathbb{C}$, the function $Y_{k,\varepsilon}$ is the so-called Weber function, defined by,

$$Y_{k,\varepsilon}(z) = \partial_\varepsilon^k Y_{0,\varepsilon}(z) \tag{4.10}$$

where $Y_{0,\varepsilon}$ is the unique entire function with respect to ε and z, solution of the Weber equation,

$$Y_{0,\varepsilon}'' + (\frac{1}{2} - \varepsilon - \frac{z^2}{4})Y_{0,\varepsilon} = 0 \tag{4.11}$$

such that, for $\varepsilon > 0$, one has,

$$Y_{0,\varepsilon}(z) \sim \frac{\sqrt{2\pi}}{\Gamma(\varepsilon)} e^{z^2/4} z^{\varepsilon - 1} \qquad (z \to +\infty). \tag{4.12}$$

As it is shown in [Pe], Theorem 4.3, a resummation of (4.7) is possible up to an error of order $\mathcal{O}(h^\infty e^{-\varphi/h})$.

Now, since φ is not C^∞ (but only C^1) near $x^{(1)}$, we need to find some generalization of lemma 4.1. For technical reasons, in the rest of this section we prefer to work with the *right* semiclassical quantization of symbols, that we denote by Op_h^R.

For $\nu_0 > 0$ and $g \in C^\infty(\mathbb{R}^{2n}\ ; \mathbb{R}_+)$, we denote by $S_{\nu_0}(g(x,\xi))$ the set of (possibly h-dependent) functions $p \in C^\infty(\mathbb{R}^{2n})$ that extend to holomorphic functions with respect to ξ in the strip,

$$\mathcal{A}_{\nu_0} := \{(x,\xi) \in \mathbb{R}^n \times \mathbb{C}^n\ ;\ |\mathrm{Im}\,\xi| < \nu_0\},$$

and such that, for all $\alpha \in \mathbb{N}^{2n}$, one has,

$$\partial^\alpha p(x,\xi) = \mathcal{O}(g(x,\mathrm{Re}\xi)), \tag{4.13}$$

uniformly with respect to $(x,\xi)\in\mathcal{A}_{\nu_0}$ and $h>0$ small enough. We also denote by $S_0(g)$ the analogous space of smooth symbols obtained by substituting $\mathbb{R}^{2n}$ to $\mathcal{A}_{\nu_0}$, and "smooth" to "holomorphic".
We first show,

Lemma 4.2. *Let $\nu_0>0$, $m\in\mathbb{R}$, $p=p(x,\xi)\in S_{\nu_0}(\langle\xi\rangle^m)$, and let $\phi=\phi(x)$ be a real bounded Lipschitz function on $\mathbb{R}^n$ such that*

$$\|\nabla\phi(x)\|_{L^\infty}<\nu_0;$$

Let also $a=a(x\,;h)\in C^\infty(\mathbb{R}^n)$ be such that, for all $\alpha\in\mathbb{N}^n$,

$$(hD_x)^\alpha a(x\,;h)=\mathcal{O}(e^{-\phi(x)/h}),$$

uniformly with respect to h small enough and $x\in\mathbb{R}^n$. Then,

$$\left(\mathrm{Op}_h^R(p)a\right)(x\,;h)=\mathcal{O}(e^{-\phi(x)/h})$$

uniformly with respect to h small enough and $x\in\mathbb{R}^n$.

Proof. - We write,

$$e^{\phi(x)/h}\mathrm{Op}_h^R(p)a(x\,;h)=\frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h+\phi(x)/h}p(y,\xi)a(y\,;h)dyd\xi, \tag{4.14}$$

and, following [Sj], we make the change of contour of integration in ξ,

$$\mathbb{R}^n\ni\xi\mapsto\xi+i\nu_1\frac{x-y}{|x-y|}, \tag{4.15}$$

where $\|\nabla\phi(x)\|_{L^\infty}<\nu_1<\nu_0$. We obtain,

$$\begin{aligned}&e^{\phi(x)/h}\mathrm{Op}_h^R(p)a(x\,;h)\\&=\frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h}p\left(y,\xi+i\nu_1\frac{x-y}{|x-y|}\right)\theta(x,y\,;h)dyd\xi,\end{aligned} \tag{4.16}$$

with,

$$\theta(x,y\,;h)=a(y\,;h)e^{(\phi(x)-\nu_1|x-y|)/h}=\mathcal{O}(e^{\phi(x)-\phi(y)-\nu_1|x-y|/h}).$$

Therefore,

$$\theta(x,y\,;h)=\mathcal{O}(e^{-\delta|x-y|/h}), \tag{4.17}$$

with $\delta=\nu_1-\|\nabla\phi\|_{L^\infty}>0$.
Then, in the case $m<-n$, the result follows immediately from (4.16)-(4.17) (and standard estimates on oscillatory integrals). In the general case, we just write,

$$\mathrm{Op}_h^R(p)=\mathrm{Op}_h^R(p)(2\nu_0-h^2\Delta_x)^{-k}(2\nu_0-h^2\Delta_x)^k, \tag{4.18}$$

with k integer large enough (e.g. $k = 1+|[m]|+n$), and, since $\mathrm{Op}_h^R(p)(2\nu_0 - h^2\Delta_x)^{-k}$ is a semiclassical pseudodifferential operators with (h-dependent) symbol in $S_{\nu_0}(\langle\xi\rangle^{m-2k}) \subset S_{\nu_0}(\langle\xi\rangle^{-n-1})$, the result follows by applying the previous case with a replaced by $(2\nu_0 - h^2\Delta_x)^k a$. $\square$

Now, in view of defining a formal pseudodifferential calculus acting on expressions such as (4.7), for $j = 1, ..., n$ and $x \in \omega_1$, we set,

$$A_j(x) := \begin{pmatrix} \frac{\partial\varphi_2(x)}{\partial x_j} & 0 \\ 0 & \frac{\partial\varphi_1(x)}{\partial x_j} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}). \tag{4.19}$$

Then, for any $k \geq 0$, we have (see [Pe] formula (4.18)),

$$\begin{aligned}(hD_{x_j} - iA_j(x)) &\begin{pmatrix} Y_{k,0}\left(\frac{z(x)}{\sqrt{h}}\right) \\ Y_{k,1}\left(\frac{z(x)}{\sqrt{h}}\right) \end{pmatrix} e^{-\psi(x)/h} \\ &= \frac{\sqrt{h}}{i}\left(\partial_{x_j} z(x)\right) \begin{pmatrix} kY_{k-1,1}\left(\frac{z(x)}{\sqrt{h}}\right) \\ Y_{k,0}\left(\frac{z(x)}{\sqrt{h}}\right) \end{pmatrix} e^{-\psi(x)/h}.\end{aligned} \tag{4.20}$$

If a and b are (scalar) formal symbols of the type (4.9), and $k \in \mathbb{N}$, we set,

$$I_k(a,b)(x\,;\,h) = a(x\,;\,h)Y_{k,0}\left(\frac{z(x)}{\sqrt{h}}\right) + b(x\,;\,h)Y_{k,1}\left(\frac{z(x)}{\sqrt{h}}\right), \tag{4.21}$$

and we make act any diagonal matrix-valued function $B(x) = \mathrm{diag}(B_1(x), B_2(x)) \in \mathcal{M}_2(\mathbb{R})$ on $I_k(a,b)(x\,;h)$ by setting,

$$B(x)I_k(a,b)(x\,;h) := I_k(B_1 a, B_2 b)(x\,;h). \tag{4.22}$$

(this is possible since the $Y_{k,0}$ and $Y_{k,1}$ are linearly independent). Then, using (4.20), for all $j = 1, ..., n$ we have,

$$\begin{aligned}&(hD_{x_j} - iA_j(x))\left(I_k(a,b)e^{-\psi/h}\right) \\ &\quad = \left(I_k(hD_{x_j}a + \sqrt{h}bD_{x_j}z\ ,\ hD_{x_j}b) + I_{k-1}(0\ ,\ k\sqrt{h}aD_{x_j}z)\right)e^{-\psi/h}.\end{aligned} \tag{4.23}$$

For $A(x) = (A_1(x), ..., A_n(x)) \in (\mathcal{M}_2(\mathbb{R}))^n$ and $\alpha \in \mathbb{N}^n$, we also use the notation,

$$A(x)^\alpha = A_1(x)^{\alpha_1}...A_n(x)^{\alpha_n} \in \mathcal{M}_2(\mathbb{R}), \tag{4.24}$$

and we identify any $\xi \in \mathbb{R}^n$ with $(\xi_1\mathcal{I}_2, ..., \xi_n\mathcal{I}_2) \in (\mathcal{M}_2(\mathbb{R}))^n$.
Then, we have,

Lemma 4.3. *Let $\nu_0 > \sup_{x\in\omega_1}\min(\sqrt{V_1(x)}, \sqrt{V_2(x)})$ and $m \in \mathbb{R}$. Then, for any $B = B(x,\xi) = \mathrm{diag}(B_1(x,\xi), B_2(x,\xi)) \in \mathcal{M}_2\left(S_{\nu_0}(\langle\xi\rangle^m)\right)$, $k \geq 0$, a*

and b in $C_0^\infty(\omega_1)$, and $\alpha \in \mathbb{N}^n$, one has,

$$
\mathrm{Op}_h^R\left(B(x,\xi)\left(\xi - iA(x)\right)^\alpha\right)\left(I_k(a,b)e^{-\psi(x)/h}\right) = \mathcal{O}\left(|\ln h|^k h^{|\alpha|/2} e^{-\varphi(x)/h}\right), \tag{4.25}
$$

where the estimates holds uniformly for h small enough and $x \in \mathbb{R}^n$.

Proof. We prove it by induction on $|\alpha|$, being careful to the fact that we do not have at disposal a symbolic calculus similar to that of the usual pseudodifferential operators (because of the special kind of action of the diagonal matrices on $I_k(a,b)$, which actually does not commute with the oscillatory integrations). We first notice that, by [Pe]-lemma 4.6, for $\beta \in \mathbb{N}^n$ and $j \in \{0,1\}$, one has,

$$
(hD_x)^\beta\left(Y_{k,j}\left(\frac{z(x)}{\sqrt{h}}\right)e^{-\psi(x)/h}\right) = \mathcal{O}\left(|\ln h|^k e^{-\varphi(x)/h}\right). \tag{4.26}
$$

As a consequence, the result for $\alpha = 0$ follows directly from Lemma 4.2. Now, assume it is true for $|\alpha| \le N$ ($N \in \mathbb{N}$ fixed arbitrarily), and let $\gamma \in \mathbb{N}^n$, $|\gamma| = 1$. Then, for $|\alpha| \le N$, we have,

$$
\begin{aligned}
&\mathrm{Op}_h^R\left(B(x,\xi)\left(\xi - iA(x)\right)^{\alpha+\gamma}\right)I_k(a,b)e^{-\psi(x)/h} \\
&\quad = \frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h}F_\alpha(y,\xi)\left(\xi - iA(y)\right)^\gamma I_k(a,b)(y)e^{-\psi(y)/h}dyd\xi,
\end{aligned} \tag{4.27}
$$

with $F_\alpha(y,\xi) = B(y,\xi)\left(\xi - iA(y)\right)^\alpha$. Now, assuming without loss of generality that $\gamma = (1,0,...,0)$, and using the fact that,

$$
\xi_1 e^{i(x-y)\xi/h} = -hD_{y_1}(e^{i(x-y)\xi/h}),
$$

we obtain (denoting by $F_{\alpha,1}$ and $F_{\alpha,2}$ the two diagonal coefficients of F_α),

$$
\begin{aligned}
&\mathrm{Op}_h^R\left(B(x,\xi)\left(\xi - iA(x)\right)^{\alpha+\gamma}\right)I_k(a,b)e^{-\psi(x)/h} \\
&= \frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h}(hD_{y_1} - iA_1(y))I_k(F_{\alpha,1}(y,\xi)a(y), F_{\alpha,2}(y,\xi)b(y)) \\
&\qquad\qquad \times e^{-\psi(y)/h}dyd\xi,
\end{aligned}
$$

and therefore, by (4.23),

$$\begin{aligned}
&\mathrm{Op}_h^R\left(B(x,\xi)\,(\xi-iA(x))^{\alpha+\gamma}\right)I_k(a,b)e^{-\psi(x)/h}\\
&\qquad=\frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h}\Big[I_k(hD_{y_1}F_{\alpha,1}a+\sqrt{h}F_{\alpha,2}bD_{y_1}z,hD_{y_1}F_{\alpha,2}b)\\
&\qquad\qquad\qquad+I_{k-1}(0,k\sqrt{h}F_{\alpha,1}aD_{y_1}z)\Big]e^{-\psi(y)/h}dyd\xi\\
&\qquad=h\mathrm{Op}_h^R(F_\alpha)I_k(D_{y_1}a,D_{y_1}b)e^{-\psi/h}+h\mathrm{Op}_h^R(D_{y_1}F_\alpha)I_k(a,b)e^{-\psi/h}\\
&\qquad\qquad\qquad+\sqrt{h}\mathrm{Op}_h^R(\mathrm{diag}(F_{\alpha,2},0))I_k(bD_{y_1}z,0)e^{-\psi/h}\\
&\qquad\qquad\qquad+\sqrt{h}\mathrm{Op}_h^R(\mathrm{diag}(0,F_{\alpha,1}))I_{k-1}(0,kaD_{y_1}z)e^{-\psi/h}.
\end{aligned}$$

Then, applying the induction hypothesis (and using the fact that $D_{y_1}F_\alpha$ is a sum of terms of the type $B'(y,\xi)(\eta-iA(y))^\beta$ with $|\beta|=|\alpha|-1$) this gives,

$$\begin{aligned}
&\mathrm{Op}_h^R\left(B(x,\xi)\,(\xi-iA(x))^{\alpha+\gamma}\right)I_k(a,b)e^{-\psi(x)/h}\\
&=\mathcal{O}\left(|\ln h|^kh^{1+\frac{|\alpha|}{2}}+|\ln h|^kh^{1+\frac{|\alpha|-1}{2}}+|\ln h|^kh^{\frac{1+|\alpha|}{2}}+|\ln h|^{k-1}h^{\frac{1+|\alpha|}{2}}\right)e^{-\varphi/h}\\
&=\mathcal{O}\left(|\ln h|^kh^{\frac{1+|\alpha|}{2}}\right)e^{-\varphi/h}
\end{aligned}$$

and the proof is complete. □

Now, for any smooth function f on $\mathbb{R}^n$, we set,

$$f(A(x))=\mathrm{diag}\left(f(\frac{\partial\varphi_2(x)}{\partial x}),f(\frac{\partial\varphi_1(x)}{\partial x})\right)\in C^\infty(\mathbb{R}^n,\mathcal{M}_2(\mathbb{C})). \tag{4.28}$$

Then, for any $p\in S_{\nu_0}(\langle\xi\rangle^m)$ and for any $N\geq 1$, Taylor's formula gives,

$$\begin{aligned}
p(x,\xi)\mathbf{I}_2=&\sum_{|\alpha|\leq N}\frac{1}{\alpha!}\partial_\xi^\alpha p(x,iA(x))(\xi-iA(x))^\alpha\\
&+\sum_{|\alpha|=N+1}B_\alpha(x,\xi)(\xi-iA(x))^\alpha,
\end{aligned}\tag{4.29}$$

where $\mathbf{I}_2$ is the 2×2 identity matrix, and the B_α's are in $\mathcal{M}_2\left(S_{\nu_0}(\langle\xi\rangle^m)\right)$. In particular, using Lemma 4.3, for any a and b in $C_0^\infty(\omega_1)$, we obtain,

$$\begin{aligned}
&\mathrm{Op}_h^R(p)\left(I_k(a,b)e^{-\psi/h}\right)\\
&\quad=\sum_{|\alpha|\leq N}\frac{1}{\alpha!}\mathrm{Op}_h^R\left(\partial_\xi^\alpha p(x,iA(x))(\xi-iA(x))^\alpha\right)\left(I_k(a,b)e^{-\psi(x)/h}\right)\\
&\qquad\qquad+\mathcal{O}(h^{N/2}e^{-\varphi/h})\\
&\quad=\sum_{\substack{|\alpha|\leq N\\ \beta\leq\alpha}}\frac{1}{i^{|\beta|}\beta!(\alpha-\beta)!}\mathrm{Op}_h^R\left(\partial_\xi^\alpha p(x,iA(x))A(x)^\beta\xi^{\alpha-\beta}\right)\left(I_k(a,b)e^{-\psi(x)/h}\right)\\
&\qquad\qquad+\mathcal{O}(h^{N/2}e^{-\varphi/h}),
\end{aligned}$$

and thus, writing down as before the corresponding oscillatory integral, in the same way we deduce,

$$\begin{aligned}&\mathrm{Op}_h^R(p)\left(I_k(a,b)e^{-\psi/h}\right)\\&\quad=\sum_{\substack{|\alpha|\leq N\\ \beta\leq\alpha}}\frac{1}{i^{|\beta|}\beta!(\alpha-\beta)!}(hD_x)^{\alpha-\beta}\left[A(x)^\beta\partial_\xi^\alpha p(x,iA(x))I_k(a,b)e^{-\psi/h}\right]\\&\qquad\qquad\qquad\qquad\qquad\qquad+\mathcal{O}(h^{N/2}e^{-\varphi/h}).\end{aligned}\tag{4.30}$$

Now, for $M\in\mathbb{Z}$ and $\Omega\subset\mathbb{R}^n$ open, we consider the space of sequences of formal symbols,

$$\begin{aligned}S^M(\omega_1):=\{a=(a_k)_{k\in\mathbb{N}}\,;a_k(x,h)=\sum_{l=-M}^{\infty}\sum_{m=0}^{l}h^l(\ln h)^m\gamma_k^{l,m}(x)\,;\\ \gamma_k^{l,m}\in C^\infty(\omega_1)\}.\end{aligned}$$

and, for $a,b\in S^M(\omega_1)$, we set,

$$I(a,b)e^{-\psi/h}:=\sum_{k\geq 0}h^kI_k(a_k,\sqrt{h}b_k)e^{-\psi/h}.\tag{4.31}$$

Using (4.23), we see that, for $j=1,...,n$, the action of $\left(hD_{x_j}-iA_j(x)\right)$ on such formal series satisfies,

$$\left(hD_{x_j}-iA_j(x)\right)I(a,b)e^{-\psi/h}=I(L_j(a,b))e^{-\psi/h},\tag{4.32}$$

where L_j is the operator,

$$\begin{aligned}L_j\ :S^M\times S^M\quad&\to\quad S^{M-1}\times S^{M-1}\\(a,b)\quad&\mapsto\quad(\widetilde{a}^j,\widetilde{b}^j)\end{aligned}$$

defined by,

$$\begin{aligned}\widetilde{a}_k^j&=hD_{x_j}a_k+hb_kD_{x_j}z\,;\\ \widetilde{b}_k^j&=hD_{x_j}b_k+(k+1)ha_{k+1}D_{x_j}z,\end{aligned}\tag{4.33}$$

($k\in\mathbb{N}$). In particular, using the notations $L=(L_1,...,L_n)$ and $L^\alpha=L_1^{\alpha_1}...L_n^{\alpha_n}$, for all $\alpha\in\mathbb{N}^n$, we have,

$$L^\alpha\ \text{ maps }\ S^M(\omega_1)\times S^M(\omega_1)\ \text{ into }\ S^{M-|\alpha|}(\omega_1)\times S^{M-|\alpha|}(\omega_1).\tag{4.34}$$

We also make naturally act any smooth diagonal $\mathcal{M}_2(\mathbb{C})$-valued function $B(x)=\mathrm{diag}(B_1(x),B_2(x))$ on $S^M\times S^M$ by setting,

$$B(a,b)=(B_1a,B_2b),\tag{4.35}$$

and we define the formal action of a pseudodifferential operator with symbol $p\in S_{\nu_0}(\langle\xi\rangle^m)$, on expressions of the type $I(a,b)e^{-\psi/h}$, by the formula,

$$\begin{aligned}\mathrm{Op}_h^F(p)\left(I(a,b)e^{-\psi/h}\right)\\ =\sum_{\substack{\alpha\in\mathbb{N}^n\\ \beta\leq\alpha}}\frac{1}{i^{|\beta|}\beta!(\alpha-\beta)!}I\left((iA(x)+L)^{\alpha-\beta}A(x)^{\beta}\partial_\xi^\alpha p(x,iA(x))(a,b)\right)e^{-\psi/h}.\end{aligned}\tag{4.36}$$

Then, in view of Lemma 4.3 and (4.30), we immediately obtain,

Proposition 4.4. *Let $a,b\in S^M(\omega_1)$ and denote by $\widetilde{I}(a,b)e^{-\psi/h}$ any resummation of $I(a,b)e^{-\psi/h}$ up to a $\mathcal{O}(h^\infty e^{-\varphi/h})$-error term. Then, for any $\chi\in C_0^\infty(\omega_1)$, the quantity $\mathrm{Op}_h^R(p)\left(\chi\widetilde{I}(a,b)e^{-\psi/h}\right)$ is a resummation of $\mathrm{Op}_h^F(p)\left(I(\chi a,\chi b)e^{-\psi/h}\right)$, up to a $\mathcal{O}(h^\infty e^{-\varphi/h})$-error term.*

In particular, the operator P naturally acts (up to $\mathcal{O}(h^\infty e^{-\varphi/h})$-error terms) on expressions of the type,

$$w_2=\begin{pmatrix} I(h\alpha_1,\beta_1)\\ I(\alpha_2,h\beta_2)\end{pmatrix}e^{-\psi/h},\tag{4.37}$$

where $\alpha_j=(\alpha_{j,k})_{k\geq0}$ and $\beta_j=(\beta_{j,k})_{k\geq0}$ are in $S^0(\omega_1)$ $(j=1,2)$.

Writing down the equation $\widetilde{P}w_2=\rho_1w_2$, setting,

$$\alpha_{j,k}=\sum_{l\geq0}\sum_{m=0}^{l}h^l(\ln h)^m\alpha_{j,k}^{l,m}(x),$$

and the analog formula for $\beta_{j,k}$, and identifying the coefficients of $h^l(\ln h)^m$ for $0\leq m\leq l\leq1$, we find (denoting by $p=\begin{pmatrix}p_1+hr_{1,1} & hr_{1,2}\\ hr_{2,1} & p_2+hr_{2,2}\end{pmatrix}$ the right-symbol of P),

$$\begin{aligned}&p_1(x,i\nabla\varphi_2)\alpha_{1,0}^{0,0}+r_{1,2}(x,i\nabla\varphi_2)\alpha_{2,0}^{0,0}+\Big[\frac{1}{i}\nabla_\xi p_1(x,i\nabla\varphi_1)(\nabla z)\\ &\qquad+\frac12\langle(\mathrm{Hess}_\xi p_1)(x,i\nabla\varphi_1)\nabla z,\nabla(\varphi_2-\varphi_1)\rangle\Big]\beta_{1,0}^{0,0}=0;\end{aligned}\tag{4.38}$$

$$\begin{aligned}&[\partial_\xi p_1(x,i\nabla\varphi_1)D_x-i(\nabla_x.\nabla_\xi p_1)(x,i\nabla\varphi_1)\\ &\qquad+r_{1,1}(x,i\nabla\varphi_1)-\rho_1]\,\beta_{1,0}^{0,0}=0;\end{aligned}\tag{4.39}$$

$$\begin{aligned}&p_2(x,i\nabla\varphi_1)\beta_{2,0}^{0,0}+r_{2,1}(x,i\nabla\varphi_1)\beta_{1,0}^{0,0}+\Big[\frac{1}{i}\partial_\xi p_2(x,i\nabla\varphi_2)(\nabla z)\\ &\qquad+\frac12\langle(\mathrm{Hess}_\xi p_2)(x,i\nabla\varphi_2)\nabla z,\nabla(\varphi_1-\varphi_2)\rangle\Big]\alpha_{2,1}^{0,0}=0;\end{aligned}\tag{4.40}$$

$$\begin{aligned}&(\partial_\xi p_2(x,i\nabla\varphi_2)D_x-i(\nabla_x.\nabla_\xi p_2)(x,i\nabla\varphi_2)\\ &\qquad+r_{2,2}(x,i\nabla\varphi_2)-\rho_1)\,\alpha_{2,0}^{0,0}=0;\end{aligned}\tag{4.41}$$

(Here we also have used the fact that $\rho \sim \sum_{k\geq 1} h^k \rho_k$ as $h \to 0$.) Identifying the other coefficients, one obtains a series of equations that (in a way similar to [Pe]-section 4) can be solved in $\mathcal{V}_1$ (possibly after having shrunk it a little bit around $x^{(1)}$), and in such a way that one also has,

$$\widetilde{w}_2 - \widetilde{w}_1 = \mathcal{O}(h^\infty e^{-\varphi/h}) \quad \text{locally uniformly in} \quad \mathcal{V}_1 \cap \{V_2 < V_1\}. \tag{4.42}$$

where w_1 is defined in (4.1), and $\widetilde{w}_1$, $\widetilde{w}_2$ are resummations of w_1 and w_2. Among other things, this implies,

$$\alpha_{2,0}^{0,0} = a_{2,0} \quad \text{in} \quad \mathcal{V}_1 \cap \{V_2 < V_1\}. \tag{4.43}$$

Moreover, we see on (4.39) and (4.41) that $\beta_{1,0}^{0,0}$ (respectively $\alpha_{2,0}^{0,0}$) are solutions of a differential equation of order 1 on each integral curve of the real vector field $\nabla\varphi_1(y).\nabla_y$ (respectively $\nabla\varphi_2(y).\nabla_y$). In particular, because of the ellipticity of $a_{2,0}$, we deduce from (4.41) and (4.43) that we have,

$$\alpha_{2,0}^{0,0} \quad \text{never vanishes in } \mathcal{V}_1. \tag{4.44}$$

Now, Assumption 6 implies that, if $\gamma \in G_0$, then,

$$r_{1,2}(x, i\nabla\varphi_2) \neq 0 \quad \text{on } \mathcal{V}_1. \tag{4.45}$$

Since $p_1(y, i\nabla\varphi_2) = p_1(y, i\nabla\varphi_1) = 0$ on $\omega_1 \cap \partial\Omega$, we deduce from (4.38) and (4.44) that, if $\gamma \in G_0$, then $\beta_{1,0}^{0,0}$ does not vanish on $\omega_1 \cap \partial\Omega$. As before, because of (4.39) (and the fact that $R(x, hD_x)$ is formally selfadjoint), this implies,

$$\text{If } \gamma \in G_0, \text{ then, } \beta_{1,0}^{0,0} \quad \text{never vanishes in } \mathcal{V}_1. \tag{4.46}$$

4.3. **In the island, outside the cirque.** Now, we look at what happens on $\gamma^{(2)}$, and, at first, near $x^{(1)}$. Using the asymptotics of $Y_{k,\varepsilon}(z/\sqrt{h})$ given in [Pe]-section 4, one also finds that, in $\mathcal{V}_1 \cap \{V_1 < V_2\}$, w_2 can be formally identified with,

$$w_3(x,h) = \sqrt{2\pi h} \begin{pmatrix} b_1(x,h) \\ hb_2(x,h) \end{pmatrix} e^{-\varphi(x)/h} \tag{4.47}$$

where b_1, b_2 are symbols of the form,

$$b_j(x;h) = \sum_{l\geq 0} \sum_{m=0}^{l} h^l (\ln h)^m b_j^{l,m}(x) \tag{4.48}$$

($j = 1, 2$), with $b_j^{l,m} \in C^\infty(\mathcal{V}_1 \cap \{V_1 < V_2\})$, in the sense that, for any resummations $\widetilde{w}_2$ and $\widetilde{w}_3$ of w_2 and w_3, one has,

$$\widetilde{w}_2 - \widetilde{w}_3 = \mathcal{O}(h^\infty e^{-\varphi/h}) \quad \text{locally uniformly in} \quad \Omega \cap \Gamma_+. \tag{4.49}$$

Moreover, one also has,

$$b_1^{0,0} = \beta_{1,0}^{0,0} \tag{4.50}$$

which, by (4.46), shows that, when $\gamma \in G_0$, b_1 is elliptic in $\mathcal{V}_1 \cap \{V_1 < V_2\}$.

Since $p_2(x, i\nabla\varphi(x)) \neq 0$ in $\{V_1 < V_2\}$, we can formally solve the equation $Pw_3 = \rho_1 w_3$, and we see again that b_1 and b_2 can be continued along the integral curves of $\nabla\varphi$, as long as these curves stay inside $\{V_1 < V_2\}$ and φ_1 does not develop caustics. In particular, they can be continued in a neighborhood $\mathcal{N}_2$ of $\gamma^{(2)}$, and the continuation of b_1 remains elliptic in Ω_2.

Clearly, the previous steps can be repeated near $x^{(2)}$, $\gamma^{(3)}$, etc... (in the case $N_\gamma \geq 3$), up to finally reach $\gamma^{(N_\gamma+1)}$, obtaining in that way (after having pasted everything in a standard way by using a partition of unity) a function $\mathbf{w}(x,h)$, smooth on a neighborhood $\mathcal{N}(\gamma)$ of γ in $\ddot{O}$, and satisfying,

$$(P - \rho_1)\mathbf{w} = \mathcal{O}(h^\infty e^{-\varphi/h}),$$

locally uniformly in $\mathcal{N}(\gamma)$. Moreover, $\mathcal{N}(\gamma)$ can be decomposed into,

$$\mathcal{N}(\gamma) = \mathcal{N}_1 \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{N_\gamma} \cup \mathcal{N}_{N_\gamma+1},$$

where, for all j, $\mathcal{V}_j$ is a neighborhood of $x^{(j)}$ and $\mathcal{N}_j$ is a neighborhood of $\gamma^{(j)}$, in such a way that, in each $\mathcal{N}_j$, $\mathbf{w}$ admits a WKB asymptotics of the form,

$$\mathbf{w}(x;h) \sim h^{\frac{j-1}{2}} \begin{pmatrix} h^{\frac{1-(-1)^j}{2}} a_1^{(j)}(x,h) \\ h^{\frac{1+(-1)^j}{2}} a_2^{(j)}(x,h) \end{pmatrix} e^{-\varphi(x)/h}, \tag{4.51}$$

where $a_1^{(j)}$ and $a_2^{(j)}$ are symbols of the same form as in (4.48), and $a_1^{(j)}$ is elliptic if j is even, while $a_2^{(j)}$ is elliptic if j is odd (in particular, $a_1^{(N_\gamma+1)}$ is elliptic). On the other hand, in each $\mathcal{V}_j$, $\mathbf{w}$ can be representated by means of the Weber function, in a way similar to that of (4.7).

4.4. **At and after the boundary of the island.** Let us denote by

$$x_\gamma \in \gamma \cap \partial\ddot{O},$$

the point of type 1 where γ touch the boundary of the island. When $x \in \gamma \cap \ddot{O}$ is close enough to x_γ, we know from the previous subsection that the asymptotic solution $\mathbf{w}$ is of the form,

$$\mathbf{w}(x;h) \sim h^{\frac{N_\gamma}{2}} \begin{pmatrix} b_1(x,h) \\ hb_2(x,h) \end{pmatrix} e^{-\varphi(x)/h}, \tag{4.52}$$

where b_1, b_2 are smooth symbols on $\mathcal{N}_{N_\gamma+1}$, of the same form as in (4.48), and b_1 is elliptic. Moreover, as x approaches x_γ, b_1 and b_2 (together with φ) develop singularities on some set $\mathcal{C}$ (called the caustic set). However, following an idea of [HeSj2], we can represent $h^{-\frac{N_\gamma}{2}}e^{S/h}w$ in the integral (Airy) form,

$$I[c_1, c_2](x,h) = h^{-1/2}\int_{\gamma(x)} \begin{pmatrix} c_1(x', \xi_n, h) \\ hc_2(x', \xi_n, h) \end{pmatrix} e^{-(x_n\xi_n+g(x',\xi_n))/h} d\xi_n, \tag{4.53}$$

where we have used local Euclidean coordinates $(x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$ centered at $\gamma\cap\partial\ddot{O}$, such that $V_1(x) = -C_0x_n+\mathcal{O}(x^2)$ near this point. For x in $\ddot{O}$ close to $\gamma\cap\partial\ddot{O}$, the phase function $\xi_n \mapsto x_n\xi_n + g(x', \xi_n)$ admits two real critical points that are close to 0. Then, choosing conveniently the x-dependent interval $\gamma(x)$, the steepest descent method at one of these points gives us the asymptotic expansion of $I[c_1, c_2]$. Comparing this with the symbols b_1 and b_2, one can determine c_1 and c_2 so that the asymptotic expansion of $h^{-\frac{N_\gamma}{2}}e^{S/h}w$ coincides with that of $I[c_1, c_2]$ in $\ddot{O}$. In particular, when $\gamma \in G_0$, one finds that c_1 remains elliptic near 0.

At this point, since we did not assume any analyticity of the potentials near $\ddot{O}$, we have to follow the methods of [FLM] where a similar situation is considered. Indeed, following the constructions of [FLM], Section 4 (that are made in the scalar case, but can be generalized without problem to our vectorial case), we see that there exists a constant $\delta > 0$ such that, for any $N \geq 1$, one can construct a (vectorial) function w_N, smooth on the set,

$$\mathcal{W}_N(\gamma) := \{|x - x_\gamma| < \varepsilon\} \cap \{\mathrm{dist}(x, \ddot{O}) < 2(Nk)^{2/3}\} \tag{4.54}$$

with $\varepsilon > 0$ small enough (recall from (3.2) that $k = |h \ln h|$), such that (see [FLM], Propositions 4.5 and 4.6),

- $(P - \rho_1)w_N = \mathcal{O}(h^{\delta N}e^{-\mathrm{Re}\,\widetilde{\varphi}_N/h})$ uniformly in $\mathcal{W}_N(\gamma)$;
- For any $\alpha \in \mathbb{Z}_+^n$, there exists $m_\alpha \geq 0$ independent of N such that,
$$\partial_x^\alpha w_N = \mathcal{O}(h^{-m_\alpha}e^{-\mathrm{Re}\,\widetilde{\varphi}_N/h})$$
uniformly in $\mathcal{W}_N(\gamma)$;
- w_N can be represented by an integral of the form (4.53) (with $\gamma(x) = \gamma_N(x)$ depending on N) in all of $\mathcal{W}_N(\gamma)$;
- $w_N = \mathbf{w}$ in $\mathcal{N}_{N_\gamma+1} \cap \mathcal{W}_N(\gamma)$;
- For any large enough L, there exist $C_L > 0$ and $\delta_L > 0$, both independent of N such that, uniformly in $\mathcal{W}_N(\gamma) \cap \{\mathrm{dist}(x, \ddot{O}) \geq$

$(Nk)^{2/3}\}$, one has,

$$\begin{aligned}(4.55)\ & w_N(x,h)\\ & = h^{\frac{N_\gamma}{2}}\left(\sum_{\substack{\ell=0\\0\leq m\leq \ell}}^{L+[Nk/C_Lh]} h^\ell(\ln h)^m \begin{pmatrix} f_{1,N}^{\ell,m}(x)\\ hf_{2,N}^{\ell,m}(x)\end{pmatrix} + \mathcal{O}(h^{\delta_L N}+h^L)\right)e^{-\widetilde{\varphi}_N(x)/h},\end{aligned}$$

as $h\to 0$, with $f_{1,N}^{\ell,m}(x), f_{2,N}^{\ell,m}(x)$ independent of h, and of the form,

$$\widetilde{f}_{j,N}^{\ell,m}(x) = (\mathrm{dist}(x,\mathcal{C}))^{-3\ell/2-1/4}\beta_{j,N}^{\ell,m}(x,\mathrm{dist}(x,\mathcal{C})), \tag{4.56}$$

$(j=1,2)$ where $\beta_{j,N}^{\ell,m}$ is smooth near $(x_\gamma,0)$, and $\beta_1^{\ell,m}(x_\gamma,0)\neq 0$ in the case $\gamma\in G_0$.

Here, $\widetilde{\varphi}_N$ is a (complex-valued) C^1 function on $\mathcal{W}_N(\gamma)$, smooth on $\mathcal{W}_N(\gamma)\backslash\mathcal{C}$, such that (see [FLM], Lemma 4.1),

- $\widetilde{\varphi}_N = \varphi + \mathcal{O}(h^\infty)$ uniformly in $\mathcal{N}_{N_\gamma+1}\cap\mathcal{W}_N(\gamma)$;
- $(\nabla\widetilde{\varphi}_N)^2 = V_1(x)+\mathcal{O}(h^\infty)$ uniformly in $\mathcal{W}_N(\gamma)$;
- There exists $\varepsilon(h)=\mathcal{O}(h^\infty)$ real, such that, for $x\in\mathcal{W}_N(\gamma)\backslash\ddot{O}$, one has,

$$\mathrm{Re}\ \widetilde{\varphi}_N(x)\geq S-\varepsilon(h); \tag{4.57}$$

- One has,

$$\mathrm{Im}\ \nabla\varphi_N(x) = -\nu_N(x)\sqrt{\mathrm{dist}(x,\mathcal{C})}\ \nabla\mathrm{dist}(x,\mathcal{C})+\mathcal{O}(\mathrm{dist}(x,\mathcal{C})),$$

uniformly with respect to $h>0$ small enough and $x\in\mathcal{W}_N(\gamma)\backslash\ddot{O}$, with $\nu_N(x)\geq\delta$.

The previous results show that we can extend $\mathbf{w}$ by taking w_N in $\mathcal{W}_N(\gamma)$, and we obtain in that way a function $\mathbf{w}_N$ smooth on $\mathcal{N}(\gamma)\cup\mathcal{W}_N(\gamma)$, such that $(P-\rho_1)\mathbf{w}_N = \mathcal{O}(h^{\delta N}e^{-\mathrm{Re}\,\widetilde{\varphi}/h})$ uniformly in $\mathcal{N}(\gamma)\cup\mathcal{W}_N(\gamma)$. Note that, thanks to Assumption 4, the number N_γ is constant on each connected component of Γ.

5. Agmon estimates

5.1. **Preliminaries.** In order to perform Agmon estimates in the same spirit as in [HeSj1], we need some preliminary results because of the fact that we have to deal with pseudodifferential operators (and not only Schrödinger operators). For this reason, we prefer to work with C^∞ weight functions (instead of Lipschitz ones), and the idea is to take h-dependent regularizations of Lipschitz weights.

At first, we need,

Proposition 5.1. *Let* $\nu_0 > 0$, $m \geq 0$, $a = a(x,\xi) \in S_{\nu_0}(\langle\xi\rangle^{2m})$. *For* $h > 0$ *small enough, let also* $\Phi_h \in C^\infty(\mathbb{R}^n)$ *real valued, such that,*

$$\sup |\nabla\Phi_h| < \nu_0, \tag{5.1}$$

and, for any multi-index $\alpha \in \mathbb{N}^n$ *with* $|\alpha| \geq 2$,

$$\partial^\alpha \Phi_h(x) = \mathcal{O}\left(h^{1-|\alpha|}\right), \tag{5.2}$$

uniformly for $x \in \mathbb{R}^n$ *and* $h > 0$ *small enough. Then, for any* $\widetilde{\Sigma} \subset \mathbb{R}^n$ *with dist*$(\Sigma, \mathbb{R}^n \backslash \widetilde{\Sigma}) > 0$, *the operator* $e^{\Phi_h/h} A e^{-\Phi_h/h} := e^{\Phi_h/h}\mathrm{Op}_h^W(a) e^{-\Phi_h/h}$ *satisfies,*

$$\| e^{\Phi_h/h} A e^{-\Phi_h/h} u\|_{L^2} \leq C_1 \|\langle hD_x\rangle^m u\|_{L^2} \tag{5.3}$$

uniformly for all $h > 0$ *small enough and* $u \in H^m(\mathbb{R}^n)$.

Proof. For $u \in C_0^\infty(\mathbb{R}^n)$, we write,

$$e^{\Phi/h} A e^{-\Phi/h} u(x) = \frac{1}{(2\pi h)^n} \int e^{i(x-y)\xi/h + (\Phi(x)-\Phi(y))/h} a(\frac{x+y}{2}, \xi) u(y) dy d\xi,$$

and the property (5.1) shows that we can make the change of contour of integration given by,

$$\mathbb{R}^n \ni \xi \mapsto \xi + i\Psi(x,y),$$

where $\Psi(x,y) := \int_0^1 \nabla\Phi((1-t)x + ty) dt$ (in particular, one has: $\Phi(x) - \Phi(y) = (x-y)\Psi(x,y)$). Then, denoting by Op_h the semiclassical quantization of symbols depending on $3n$ variables (see e.g. [Ma2] Section 2.5), we obtain,

$$e^{\Phi/h} A e^{-\Phi/h} = \mathrm{Op}_h \left(a(\frac{x+y}{2}, \xi + i\Psi(x,y)) \right),$$

and, using (5.2), we see that, for any $\alpha, \beta, \gamma \in \mathbb{Z}_+^n$, we have

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma \left(a(\frac{x+y}{2}, \xi + i\Psi(x,y)) \right) = \mathcal{O}(h^{-|\alpha+\beta|}\langle\xi\rangle^m). \tag{5.4}$$

Then, the results is an easy consequence of the Calderón-Vaillancourt Theorem: see, e.g., [Ma2], exercise 2.10.15. □

We also need,

Proposition 5.2. *Let* ϕ *and* V *be two bounded real-valued Lipschitz functions on* $\mathbb{R}^n$, *such that* $|\nabla\phi(x)|^2 \leq V(x)$ *almost everywhere. Let also*

$\chi_1 \in C_0^\infty(\mathbb{R}^n;[0,1])$ *supported in the ball* $\{|x|\leq 1\}$, *such that* $\int \chi_1(x)dx = 1$. *For any* $h>0$, *we set* $\chi_h(x) = h^{-n}\chi(x/h)$. *Then, the smooth function,*

$$\phi_h := \chi_h * \phi$$

(where $*$ *stands for the standard convolution) satisfies,*

- $\phi_h = \phi + \mathcal{O}(h)$ *uniformly for* $h>0$ *small enough and* $x\in\mathbb{R}^n$;
- *For all* $x\in\mathbb{R}^n$, *one has* $|\nabla\phi_h(x)|^2 \leq V(x) + h\|\nabla V\|_{L^\infty}$;
- *For all* $\alpha\in\mathbb{Z}_+^n$ *with* $|\alpha|\geq 1$, *one has* $\partial^\alpha\phi_h = \mathcal{O}(h^{1-|\alpha|})$.

The proof of this proposition is very standard and almost obvious, and we leave it to the reader. Observe that, in particular, ϕ_h satisfies the estimates (5.2).

5.2. **Agmon estimates.** As a corollary of the two previous propositions, we have,

Corollary 5.3. *Let* ϕ *and* ϕ_h *be as in Proposition 5.2, with* $V = \min(V_1,V_2)_+$. *Then, for any* $u=(u_1,u_2)\in H^2(\mathbb{R}^n)\oplus H^2(\mathbb{R}^n)$, *one has,*

$$\begin{aligned}\mathrm{Re}\,\langle e^{\phi_h/h}Pu, e^{\phi_h/h}u\rangle \geq \|h\nabla(e^{\phi_h/h}u)\|^2 + \sum_{j=1}^{2}\langle (V_j - |\nabla\phi_h|^2)e^{\phi_h/h}u_j, e^{\phi_h/h}u_j\rangle \\ -C_R h(\|e^{\phi_h/h}u\|^2 + \|h\nabla(e^{\phi_h/h}u)\|^2),\end{aligned}$$

where $C_R>0$ *is a constant that depends on* $R(x,hD_x)$, χ_1 *and* $\sup|\nabla\phi|$ *only.*

Proof. It is standard (and elementary) to show that,

$$\begin{aligned}&\mathrm{Re}\,\langle e^{\phi_h/h}(-h^2\Delta + V_j)u_j, e^{\phi_h/h}u_j\rangle \\ &\qquad = \|h\nabla(e^{\phi_h/h}u_j)\|^2 + \langle (V_j - |\nabla\phi_h|^2)e^{\phi_h/h}u_j, e^{\phi_h/h}u_j\rangle.\end{aligned}$$

Therefore, it is enough to estimate $\langle e^{\phi_h/h}R(x,hD_x)u, e^{\phi_h/h}u\rangle$. Applying Proposition 5.1 , we see that the operator $e^{\phi_h/h}R(x,hD_x)e^{-\phi_h/h}\langle hD_x\rangle^{-1}$ is uniformly bounded on L^2.
Moreover, since the constants appearing in the estimates (5.4) depend on a, α, and on the estimates on the $\partial^\beta\Phi$'s only, we see that the norm of $e^{\phi_h/h}R(x,hD_x)e^{-\phi_h/h}\langle hD_x\rangle^{-1}$ depends on r and on estimates on $\partial^\beta(\chi_h * \nabla\phi) = (\partial^\beta\chi_h) * \nabla\phi$ ($|\beta|\leq|\alpha|$) only. Since the latter depend on α, χ_1 and $\sup|\nabla\phi|$ only, the result follows. $\square$

6. Global asymptotic solution

The constructions of Section 4 can be done in a neighborhood of any minimal geodesic $\gamma \in G$, and give rise (after having pasted them together with a partition of unity) to an asymptotic solution (still denoted by $\mathbf{w}_N$) on a neighborhood of $\bigcup_{\gamma\in G}\gamma$. Now, we plan to extend this solution to a whole (h-dependent) neighborhood of $\{V_1 \geq 0\}$, by using a modified selfadjoint operator with discrete spectrum near 0.

At first, we fix $\varepsilon_0 > 0$ sufficiently small, and a cut-off function $\chi_0 \in C_0^\infty(\ddot{O};[0,1])$ such that,

$$\chi_0(x) = 1 \text{ if } V_1(x) \geq 2\varepsilon_0\,;\ \chi_0(x) = 0 \text{ if } V_1(x) \leq \varepsilon_0,$$

and we set,

$$\widetilde{V}_1 := \chi_0 V_1 + \varepsilon_0(1-\chi_0). \tag{6.1}$$

In particular, $\widetilde{V}_1$ coincides with V_1 on $\{V_1 \geq 2\varepsilon_0\}$, and we have $\widetilde{V}_1 \geq \varepsilon_0$ everywhere. Then, we define $\widetilde{P}_1 := -h^2\Delta + \widetilde{V}_1$, and we consider the selfadjoint operator,

$$\widetilde{P} = \begin{pmatrix} \widetilde{P}_1 & 0 \\ 0 & P_2 \end{pmatrix} + hR(x,hD_x). \tag{6.2}$$

By construction, for all $C > 0$ and h small enough, the spectrum of $\widetilde{P}$ is discrete in $[-Ch, Ch]$, and a straightforward adaptation of the arguments used in [HeSj1] shows that its first eigenvalue E_1 admits the same asymptotics as ρ_1 as $h \to 0_+$. We denote by $\mathbf{v}$ its first normalized eigenfunction, and by $\mathcal{N}_0 \subset \{V_1 > 2\varepsilon_0\}$ some fix neighborhood of $\bigcup_{\gamma\in G}\cap\{V_1 > 2\varepsilon_0\}$ where the asymptotic solution $\mathbf{w}_N$ is well defined. We have,

Proposition 6.1. *There exists $\theta_0 \in \mathbb{R}$ independent of h, such that, for any compact subset K of $\mathcal{N}_0$, and for any $\alpha \in \mathbb{Z}_+^n$, one has,*

$$\|e^{\varphi/h}\partial^\alpha(e^{i\theta_0}\mathbf{v} - h^{\frac{n}{4}}\mathbf{w}_N)\|_K = \mathcal{O}(h^\infty).$$

Proof. The existence of θ_0 such that $\partial^\alpha(e^{i\theta_0}\mathbf{v} - h^{\frac{n}{4}}\mathbf{w}_N) = \mathcal{O}(h^\infty)$ uniformly near 0, is a consequence of [HeSj1], Proposition 2.5, and standard Sobolev estimates. Let $\chi \in C_0^\infty(\mathcal{N}_0;[0,1])$, with $\chi = 1$ in a neighborhood of $K \cup \{0\}$. Following [HeSj1, Pe], we plan to apply Corollary 5.3 to $u := \chi(e^{i\theta_0}\mathbf{v} - h^{\frac{n}{4}}\mathbf{w}_N)$, with a suitable weight function ϕ. Let us first observe that, using

Corollary 5.3, for any $\varepsilon > 0$, one has,

$$\|e^{(1-\varepsilon)\widetilde{\varphi}/h}\langle hD_x\rangle \mathbf{v}\|_{H^1} = \mathcal{O}(1). \tag{6.3}$$

where $\widetilde{\varphi}(x) \geq \varphi(x)$ is the Agmon distance associated with $\min(\widetilde{V}_1, V_2)$ between 0 and x. Now, for $C \geq 1$ arbitrarily large, we define,

$$\phi(x) := \min(\phi_1, \phi_2),$$

where,

$$\phi_1(x) := \begin{cases} \varphi(x) - Ch\ln(\varphi(x)/h) & \text{if } \varphi(x) \geq Ch; \\ \varphi(x) - Ch\ln C & \text{if } \varphi(x) \leq Ch, \end{cases}$$

$$\phi_2(x) := \begin{cases} \inf\limits_{\chi(y)\neq 1}(1-2\varepsilon)(\varphi(y) + d(y,x)) & \text{if } x \in \mathrm{supp}\chi; \\ (1-2\varepsilon)\varphi(x) & \text{if } x \notin \mathrm{supp}\chi. \end{cases}$$

Here, $\varepsilon > 0$ is taken sufficiently small in order to have $\phi_2(x) > \varphi(x)$ when $x \in K$. Then, ϕ is Lipschitz continuous, and one has $\phi = \phi_1$ on K, and $\phi = \phi_2$ on $\mathbb{R}^n \backslash \{\chi = 1\}$. Moreover, one sees as in [Pe] (proof of Theorem 5.5) that, if we set $V := \min(V_1, V_2)$, ϕ satisfies,

$$|\nabla\phi|^2 = V \text{ in } \{\varphi \leq Ch\};$$
$$|\nabla\phi|^2 \leq V - \delta_0 Ch \text{ in } \{\varphi \geq Ch\},$$

where $\delta_0 = \inf_{x\in\mathrm{supp}\chi\,;\,x\neq 0}(V(x)/\varphi(x)) > 0$. As a consequence, by Proposition 5.2, the regularized ϕ_h of ϕ satisfies,

$$|\nabla\phi_h|^2 \leq V + h\|V\|_{L^\infty} \text{ in } \{\varphi \leq Ch\};$$
$$|\nabla\phi_h|^2 \leq V - (\delta_0 C - \|V\|_{L^\infty})h \text{ in } \{\varphi \geq Ch\}.$$

Then, choosing C sufficiently large, and setting $u := \chi(e^{i\theta_0}\mathbf{v} - h^{\frac{n}{4}}\mathbf{w}_N)$, we see that Corollary 5.3 implies,

$$\|h\nabla(e^{\phi_h/h}u)\|^2 + C'h\|e^{\phi_h/h}u\|^2_{\{\varphi\geq Ch\}} \leq \langle e^{\phi_h/h}(\widetilde{P} - E_1)u, e^{\phi_h/h}u\rangle, \tag{6.4}$$

with $C' = C'(C)$ arbitrarily large. Moreover, if $\widetilde{\chi} \in C_0^\infty(\mathcal{N}_0)$ is such that $\widetilde{\chi}\chi = \chi$, we have,

$$(\widetilde{P} - E_1)u = [\widetilde{P}, \chi]\widetilde{\chi}u + \mathcal{O}(h^\infty e^{-\varphi/h}),$$

and since $\phi_h = (1-2\varepsilon)\varphi + \mathcal{O}(h)$ on $\mathrm{supp}\nabla\chi$, $\min_{\mathrm{supp}\nabla\chi}\varphi =: \delta_1 > 0$, and, by Proposition 5.1, the operator $e^{\phi_h/h}[R, \chi]e^{-\phi_h/h}$ is uniformly bounded, we obtain (using also (6.3)),

$$\begin{aligned} \langle e^{\phi_h/h}(\widetilde{P} - E_1)u, e^{\phi_h/h}u\rangle &= \mathcal{O}\left(\|e^{(1-\varepsilon)\varphi/h}\langle hD_x\rangle\widetilde{\chi}u\|^2_{\mathrm{supp}\nabla\chi} + h\|e^{\phi_h/h}u\|^2\right) \\ &= \mathcal{O}\left(e^{-2\varepsilon\delta_1/h} + h\|e^{\phi_h/h}u\|^2\right). \end{aligned}$$

Inserting this estimate into (6.4), and taking C sufficiently large, this permits us to obtain,

$$\|h\nabla(e^{\phi_h/h}u)\|^2 + h\|e^{\phi_h/h}u\|^2 = \mathcal{O}(e^{-2\varepsilon\delta_1/h} + \|e^{\phi_h/h}u\|^2_{\{\varphi\leq Ch\}}).$$

In particular, since $\phi_h = \phi_1 + \mathcal{O}(h)$ on K, and $\phi_h = (1-2\varepsilon)\varphi \leq Ch$ on $\{\varphi \leq Ch\}$,

$$\|h^C\varphi^{-C}e^{\varphi/h}h\nabla u\|^2_K + \|h^C\varphi^{-C}e^{\varphi/h}u\|^2_K = \mathcal{O}(e^{-2\varepsilon\delta_1/h} + \|u\|^2_{\{\varphi\leq Ch\}}).$$

Therefore,

$$\|e^{\varphi/h}\nabla u\|^2_K + \|e^{\varphi/h}u\|^2_K = \mathcal{O}(h^\infty),$$

and the result follows by standard Sobolev estimates. $\square$

Now, following [FLM], Section 4.3, we observe that, if ε_0 has been taken small enough, the asymptotic solution $\mathbf{w}_N$ is $\mathcal{O}(h^{\delta N}e^{-S/h})$ uniformly on the set,

$$\{\mathrm{dist}(x, \bigcup_{\gamma\in G}\gamma) \geq \varepsilon_0\} \cap \{V_1 \leq 2\varepsilon_0\} \cap \left(\bigcup_{\gamma\in G}\mathcal{N}(\gamma)\cup\mathcal{W}_N(\gamma)\right)$$

Moreover, by (6.3), the same is true for $\mathbf{v}$ on $\{\mathrm{dist}(x, \bigcup_{\gamma\in G}\gamma) \geq \varepsilon_0\} \cap \{V_1 \leq 2\varepsilon_0\}$. Therefore, using also Proposition 6.1, we can paste together $e^{i\theta_0}\mathbf{v}$ and $h^{-n/4}\mathbf{w}_N$ in order to obtain a function $\mathbf{u}_N$ that satisfies to the properties of the following proposition (see also [FLM], Proposition 4.6):

Proposition 6.2. *There exists a function $\mathbf{u}_N$, smooth on $\ddot{O}_N := \{\mathrm{dist}(x,\ddot{O})\} < 2(Nk)^{2/3}$, such that,*

$$(P-\rho)\mathbf{u}_N = \mathcal{O}(h^{\delta N}e^{-\mathrm{Re}\,\varphi_N/h});$$
$$\partial^\alpha\mathbf{u}_N = \mathcal{O}(h^{-m_\alpha}e^{-\mathrm{Re}\,\varphi_N/h}),$$

uniformly on $\ddot{O}_N$, where $\widetilde{\varphi}_N$ is as in (4.57). Moreover, $\mathbf{u}_N$ can be written as in (4.55) in $\bigcup_{\gamma\in G}\mathcal{W}_N(\gamma)\cap\{\mathrm{dist}(x,\ddot{O}) \geq (Nk)^{2/3}\}$ (with $\beta_1^{\ell,m}(x_\gamma, 0)\neq 0$), while $\mathbf{u}_N$ is $\mathcal{O}(h^{\delta N}e^{-\mathrm{Re}\,\varphi_N/h})$ away from $\bigcup_{\gamma\in G}\mathcal{W}_N(\gamma)\cap\{x\notin\ddot{O}\}$.

7. Comparison between asymptotic and true solution

7.1. **A priori estimates.** In the same spirit as in [FLM], Theorem 2.2, we start with an a priori estimate for the resonant state of P. From now on,

we denote by $\mathbf{u}$ the out-going solution of

$$P\mathbf{u} = \rho_1 \mathbf{u}, \tag{7.1}$$

normalized in the following way: we fix some analytic distorted space (also known, more recently, as a Perfectly Matched Layer), of the form,

$$\widetilde{\mathbb{R}}^n_\theta := \{x + i\theta F(x)\,;\, x \in \mathbb{R}^n\}, \tag{7.2}$$

where $F \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $F(x) = 0$ if $|x| \leq R_0$, $F(x) = x$ for $|x|$ large enough, and where $\theta > 0$ is sufficiently small, and may also tend to 0 with h, but not too rapidly (here, we take $\theta = h|\ln h| = k$). Then, by definition, the fact that ρ_1 is a resonance of P means that Equation (7.1) admits a solution in $L^2(\widetilde{\mathbb{R}}^n_\theta)$, and here we take $\mathbf{u}$ in such a way that,

$$\|\mathbf{u}\|_{L^2(\widetilde{\mathbb{R}}^n_\theta)} = 1. \tag{7.3}$$

As before, d stands for the Agmon distance associated with the pseudometric $\min(V_1, V_2)_+ dx^2$, and we denote by $B_d(S) := \{x \in \mathbb{R}^n)\,;\, d(0,x) < S\}$ the corresponding open ball of radius $S = d(0, \partial\ddot{O})$. Then, we first have,

Proposition 7.1. *For any compact subset $K \subset \mathbb{R}^n$, there exists $N_K \geq 0$ such that,*

$$\|e^{s(x)/h}\mathbf{u}\|_{H^1(K)} = \mathcal{O}(h^{-N_K}),$$

uniformly as $h \to 0$, where $s(x) = \varphi(x)$ if $x \in B_d(S)$ and $s(x) = S$ otherwise.

Proof. The proof if very similar to that of [FLM] Theorem 2.2, with the only difference that here we have to deal with pseudodifferentilal operators, forbidding us to use Dirichlet realizations and non smooth weight functions. Instead, we modify V_1 in a way similar to (6.1), and we regularize the weights as in Proposition 5.2.

We consider a (h-dependent) cutoff function $\hat{\chi}$ such that,

$$\hat{\chi}(x) = 1 \text{ if } V_1(x) \geq 2k^{2/3}\,;\, \hat{\chi}(x) = 0 \text{ if } V_1(x) \leq k^{2/3}\,;\, \partial^\alpha \hat{\chi} = \mathcal{O}(k^{-2|\alpha|/3}),$$

and we set,

$$\hat{V}_1 := \hat{\chi} V_1 + k^{2/3}(1 - \hat{\chi}) \quad ; \quad \hat{P}_1 := -h^2\Delta + \hat{V}_1;$$

$$\hat{P} = \begin{pmatrix} \hat{P}_1 & 0 \\ 0 & P_2 \end{pmatrix} + hR(x, hD_x). \tag{7.4}$$

We denote by $\hat{E}$ the first eigenvalue of $\hat{P}$, and by $\hat{v}$ its first normalized eigenfunction. Moreover, we consider the Agmon distance $\hat{d}$ associated with the

pseudometric $\left(\min(V_1,V_2)_+ - \hat{E}\right) dx^2$, and we set $\hat{\varphi}(x) := \hat{d}(0,x)$. Then, the same proof as in [FLM], Lemma 3.1, shows the existence of a constant $C_1 > 0$ such that,

$$s(x) - C_1 k \leq \hat{\varphi}(x) \leq \varphi(x) \quad (x \in \mathbb{R}^n). \tag{7.5}$$

Moreover, an adaptation of the proof of [FLM], Lemma 3.2 (obtained by using Proposition 5.2 in order to regularize the Lipschitz weight) gives,

$$\| e^{\hat{\varphi}/h} \hat{v} \|_{H^1(\mathbb{R}^n)} = \mathcal{O}(h^{-N_0}), \tag{7.6}$$

for some $N_0 \geq 0$. Then, the result follows by considering the function $\hat{\chi}\hat{v}$, and by observing that, thanks to (7.6), one has (see [FLM], Lemma 3.3 and Formula (3.20)),

$$\| \hat{\chi}\hat{v} - \frac{1}{2i\pi} \int_\gamma (z - P_\theta)^{-1} \hat{\chi}\hat{v}\, dz \|_{H^1} = \mathcal{O}(h^{-N_1} e^{-S/h}).$$

Here, γ is the oriented complex circle $\{z \in \mathbb{C}\,;\, |z - \hat{E}| = h^2\}$, and P_θ is a convenient distortion of P. The previous estimate actually shows that the distorted $\mathbf{u}_\theta$ of $\mathbf{u}$ coincides – up to $\mathcal{O}(h^{-N_1} e^{-S/h})$ – with $\mu \hat{\chi}\hat{v}$, where μ is a complex constant satisfying $|\mu| = 1 + \mathcal{O}(e^{-\delta/h})$, for some $\delta > 0$. $\square$

Remark 7.2. *The previous proof also gives a global estimate on* $\mathbf{u}_\theta$*,*

$$\| e^{s(x)/h} \mathbf{u}_\theta \|_{H^1(\mathbb{R}^n)} = \mathcal{O}(h^{-N_1'}),$$

for some constant $N_1' \geq 0$*. See* [FLM]*, Lemma 3.3 and Formula (3.20).*

Now, we plan to give an even better a priori estimate on the difference $\mathbf{u} - \mathbf{u}_N$ near the boundary of the island. Here again, we follow the arguments given in [FLM], Section 5. For any $N \geq 1$, we set,

$$U_N := \{x \in \mathbb{R}^n\,;\, \mathrm{dist}(x, \partial \ddot{O}) < 2(Nk)^{2/3}\}.$$

We have (see [FLM], Propositions 5.1 and 5.2),

Proposition 7.3. *There exists* $N_1 \geq 0$ *and* $C \geq 1$ *such that, for any* $N \geq 1$ *large enough, one has,*

$$\| \mathbf{u} - \mathbf{u}_{CN} \|_{H^1(U_N)} \leq h^{-N_2} e^{-S/h}.$$

Proof. We just recall the main lines of the proof in [FLM]. At first, thanks to Proposition 7.1 and the particular form of $\mathbf{u}_{CN}$, we immediately see that the estimate is true on the set $\{\varphi(x) \geq S - 2k\}$. Then, we take a cutoff

function $\widetilde{\chi} \in C_0^\infty(\varphi(x) < S-k)$ such that $\widetilde{\chi} = 1$ on $\{\varphi(x) \geq S-2k\}$, and $\partial^\alpha \widetilde{\chi} = \mathcal{O}(h^{-N_\alpha})$ for some $N_\alpha \geq 0$. We also consider the Lipschitz weight,

$$\phi_N(x) = \min\left(\varphi(x) + C_1 Nk + k(S-\varphi(x))^{1/3}, S + (1-k^{1/3})(S-\varphi(x))\right),$$

and, by using Propositions 7.1 and 6.2, we see that if C is large enough, we have,

$$\|e^{\phi_N/h}(P-\rho_1)\widetilde{\chi}(\mathbf{u}-\mathbf{u}_{CN})\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^{-M_1}),$$

for some $M_1 \geq 0$ independent of N. Then, regularizing ϕ_N as in Proposition 5.2, we can perform Agmon estimates as in the proof of [FLM], Proposition 5.1, and we find,

$$\|e^{\phi_N/h}\widetilde{\chi}(\mathbf{u}-\mathbf{u}_{CN})\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^{-M_2}),$$

for some $M_1 \geq 0$ independent of N, and the result follows. □

7.2. **Propagation.** Now, we plan to prove (see [FLM], Proposition 6.1),

Theorem 7.4. *For any $L > 0$ and for any $\alpha \in \mathbb{Z}_+^n$, there exists $N_{L,\alpha} \geq 1$ such that, for any $N \geq N_{L,\alpha}$, one has,*

$$\partial_x^\alpha(\mathbf{u}-\mathbf{u}_{CN})(x,h) = \mathcal{O}(h^L e^{-S/h}) \quad \text{as} \quad h \to 0, \tag{7.7}$$

uniformly in U_N.

Proof. As in [FLM], the proof relies on three different types of microlocal propagation arguments. We fix some $\hat{x} \in \partial\ddot{O}$, and we define the Fourier-Bors-Iagolnitzer transform T (see, e.g., [Sj, Ma2]) as,

$$Tu(x,\xi;h) := \int_{\mathbb{R}^n} e^{i(x-y)\xi/h-(x-y)^2/2h}u(y)dy.$$

1) *Standard C^∞ propagation*

Since $\mathbf{u}$ is outgoing (that is, it becomes L^2 when restricted to distorted space or Perfectly Matched Layer defined in (7.2)), one can see as in [FLM], Lemma 6.2 that, if $t_0 > 0$ is large enough, then, one has,

$$T\mathbf{u}(x,\xi) = \mathcal{O}(h^\infty e^{-S/h}),$$

uniformly near $\exp(-t_0 H_{p_1})(\hat{x},0)$. Moreover, by Proposition 7.1, we know that $e^{S/h}\mathbf{u}$ remains $\mathcal{O}(h^{-N_0})$ (for some $N_0 \geq 0$) on a neighborhood of the x-projection of $\{\exp(-tH_{p_1})(\hat{x},0)\,;\, 0 < t \leq t_0\}$.

Then, the standard C^∞ propagation of the Frequency Set for the solution to a real principal type operator (see, e.g., [Ma2]) shows that the previous estimate remain valid near $\exp(-tH_{p_1})(\hat{x},0)$ for any $t>0$.

2) *Non standard propagation in h-dependent domains*

Thanks to the previous result, we can concentrate our attention to a sufficiently small neighborhood of $\hat{x}$. As before, we choose local Euclidean coordinates $(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$ centered at $\hat{x}$, such that $V_1(x)=-C_0x_n+\mathcal{O}(|x-\hat{x}|^2)$. We also set $\mu_N:=(Nk)^{-1/3}$, and we considered the modified Fourier-Bors-Iagolnitzer transform T_N defined by,

$$T_Nu(x,\xi;h):=\int_{\mathbb{R}^n}e^{i(x-y)\xi/h-(x'-y')^2/2h-\mu_N(x_n-y_n)^2/2h}u(y)dy. \tag{7.8}$$

Then, using the previous result it is elementary to show that (see [FLM], Lemma 6.3), for any (fixed) $t>0$ small enough, one has,

$$T_N\mathbf{1}_{K_1}\mathbf{u}(x,\xi)=\mathcal{O}(h^\infty e^{-S/h}),$$

uniformly near $\exp(-tH_{p_1})(\hat{x},0)$,. Here K_1 is of the form $K_1=K\backslash B_d(S)$, where K is any compact neighborhood of the closure of $\ddot{O}$. The interest of the latter property is that, as shown in [FLM], it can be propagated up h-dependent times t of order $(Nk)^{1/3}$. More precisely, setting,

$$\exp tH_{p_1}(\hat{x},0)=(x'(t),x_n(t);\xi'(t),\xi_n(t))\quad(t\in\mathbb{R}),$$

we have (see [FLM], Lemma 6.4),

Lemma 7.5. *There exists $\delta_0>0$ such that, for any $\delta\in(0,\delta_0]$, for all $N\geq1$ large enough, and for $t_{N,\delta}:=\delta^{-1}(Nk)^{1/3}$, one has,*

$$\mathbf{T}_N\mathbf{1}_{K_1}\mathbf{u}=\mathcal{O}(h^{\delta N}e^{-S/h})\ \textit{uniformly in}\ \mathcal{W}(t_N,h),$$

where,

$$\begin{aligned}\mathcal{W}_\delta(N,h):=\{&|x_n-x_n(-t_{N,\delta})|\leq\delta(Nk)^{2/3}\,,\,|\xi_n-\xi_n(-t_{N,\delta})|\leq\delta(Nk)^{1/3},\\&|x'-x'(-t_{N,\delta})|\leq\delta(Nk)^{1/3}\,,\,|\xi'-\xi'(-t_{N,\delta})|\leq\delta(Nk)^{1/3}\}.\end{aligned}$$

Proof. The proof is based on the refined exponential weighted estimates (in the same spirit as in [Ma2]) given in [FLM], Proposition 8.3 , that we apply here to the operator P_1. Since the proof is very similar to that of [FLM], Lemma 6.4, we omit the details. □

On the other hand, using the explicit form of $\mathbf{u}_{CN}$ given in (4.55), one also sees that, for any L large enough, there exists $\delta_L > 0$ such that, for any $N \geq 1$, one has (see [FLM], Lemma 6.7),

$$T_N \mathbf{1}_{K_1} \mathbf{u}_{CN} = \mathcal{O}((h^{\delta_L N} + h^L)e^{-S/h}) \text{ uniformly in } \mathcal{W}_\delta(N,h).$$

In particular, taking $N = L/\delta_L$ with $L >> 1$, we obtain a sequence $N = N_L$ along which,

$$T_N \mathbf{1}_{K_1} \mathbf{u}_{CN} = \mathcal{O}(h^{\delta_L N} e^{-S/h}) \text{ uniformly in } \mathcal{W}_\delta(N,h),$$

and with both N and $\delta_L N$ arbitrarily large.

As a consequence, along the same sequence, we also obtain,

$$T_N \mathbf{1}_{K_1}(\mathbf{u} - \mathbf{u}_{CN}) = \mathcal{O}(h^{\delta_L' N} e^{-S/h}) \text{ uniformly in } \mathcal{W}_\delta(N,h),$$

with $\delta_L' = \min(\delta, \delta_L)$.

Moreover, we see that, when $y \in U_N \cap B_d(S)$ and $x \in \Pi_x \mathcal{W}_\delta(N,h)$ (where Π_x stands for the natural projection onto the x-space), we have,

$$\mu_N(x_n - y_n)^2 + s(x) - S \geq C_\delta N k,$$

with $C_\delta > 0$ constant (and actually, $C_\delta \to \infty$ as $\delta \to 0$). Therefore, using Proposition 7.3 and the expression (7.8) of T_N, we also obtain,

$$T_N \mathbf{1}_{U_N \cap B_d(S)}(\mathbf{u} - \mathbf{u}_{CN}) = \mathcal{O}(h^{\delta N} e^{-S/h}) \text{ uniformly in } \mathcal{W}_\delta(N,h).$$

as a consequence, if we set,

$$\chi_N(x) := \chi_0\left(\frac{|x_n - \hat{x}_n|}{(Nk)^{2/3}}\right)\chi_0\left(\frac{|x' - \hat{x}'|}{(Nk)^{1/2}}\right), \tag{7.9}$$

where the function $\chi_0 \in C_0^\infty(\mathbb{R}_+;[0,1])$ verifies $\chi_0 = 1$ in a sufficiently large neighborhood of 0, and is fixed in such a way that $\chi_N(x) = 1$ in $\{|x_n - \hat{x}_n| \leq |x_n(-t_N) - \hat{x}_n| + 2\delta(Nk)^{2/3}\,;\, |x' - \hat{x}'| \leq |x'(-t_N) - \hat{x}'| + 2\delta(Nk)^{1/2}\}$ (here, t_N and δ are those of Lemma 7.5), then, the function,

$$v_N := \chi_N e^{S/h}(\mathbf{u} - \mathbf{u}_{CN}),$$

is such that,

$$T_N v_N = \mathcal{O}(h^{\delta_L' N} e^{-S/h}) \text{ uniformly in } \mathcal{W}_\delta(N,h). \tag{7.10}$$

Moreover, we have (see [FLM], Section 6.2),

$$(P - \rho_1)v_N = [P, \chi_N]e^{S/h}(\mathbf{u} - \mathbf{u}_{CN}) + \mathcal{O}(h^{\delta N}),$$

and thus, on $\{d_N(x, \mathrm{supp}\nabla\chi_N) \geq \varepsilon\} \times \mathbb{R}^n$ (where $\varepsilon > 0$ is fixed small enough and d_N is the distance associated with the metric $(Nk)^{-1}(dx')^2 + (Nk)^{-4/3}dx_n^2$),

$$T_N(P-\rho_1)v_N = \mathcal{O}(h^{\delta' N}),$$

for some $\delta' = \delta'(\varepsilon) > 0$.

3) *(Almost) standard analytic propagation*

Although we are in a region where no analytic assumption is made, a re-scaling of the problem makes appear estimates similar to those encountered in the analytic context. Indeed, setting,

$$\widetilde{h} = \widetilde{h}_N := \frac{h}{Nk} = \left(N \ln \frac{1}{h}\right)^{-1},$$

and performing the change of variable (still working in the same coordinates for which $\hat{x} = 0$),

$$\begin{aligned} x \mapsto \widetilde{x} &= (\widetilde{x}', \widetilde{x}_n) := ((Nk)^{-1/2}x', (Nk)^{-2/3}x_n);\\ \xi \mapsto \widetilde{\xi} &= (\widetilde{\xi}', \widetilde{\xi}_n) := ((Nk)^{-1/2}\xi', (Nk)^{-2/3}\xi_n) \end{aligned}$$

we see that the estimate (7.10) implies (see [FLM], Formula (6.43)),

$$T\widetilde{v}_N(\widetilde{x}, \widetilde{\xi}; \widetilde{h}_N) = \mathcal{O}(e^{-\delta'_L/2\widetilde{h}_N})$$

uniformly in the tubular domain,

(7.11)

$$\begin{aligned} \widetilde{\mathcal{W}}(\widetilde{h}) := \{ \quad & |\widetilde{x}_n - \widetilde{x}_n(-\delta^{-1})| \leq \delta\,, & & |\widetilde{\xi}_n - \widetilde{\xi}_n(-\delta^{-1})| \leq \delta,\\ & |\widetilde{x}' - \widetilde{x}'(-\delta^{-1})| \leq \delta(Nk)^{-\frac{1}{6}}\,, & & |\widetilde{\xi}' - \widetilde{\xi}'(-\delta^{-1})| \leq \delta(Nk)^{-\frac{1}{6}}\}, \end{aligned}$$

where,

$$\begin{aligned} &\widetilde{v}_N(\widetilde{x}) := (Nk)^{\frac{n-1}{4}+\frac{1}{3}} v_N((Nk)^{\frac{1}{2}}\widetilde{x}', (Nk)^{\frac{2}{3}}\widetilde{x}_n);\\ &(\widetilde{x}(\widetilde{t}), \widetilde{\xi}(\widetilde{t})) := \exp \widetilde{t} H_{\widetilde{p}_1}(0,0);\\ &\widetilde{p}_1(\widetilde{x}, \widetilde{\xi}) := (Nk)^{1/3}|\widetilde{\xi}'|^2 + \widetilde{\xi}_n^2 + W_1(\widetilde{x}, \widetilde{h});\\ &W_1(\widetilde{x}, \widetilde{h}) := (Nk)^{-2/3}V_1((Nk)^{1/2}\widetilde{x}', (Nk)^{2/3}\widetilde{x}_n) - (Nk)^{-2/3}\rho_1. \end{aligned}$$

Moreover, setting,

$$\widetilde{P} := -(Nk)^{1/3}\widetilde{h}^2\Delta_{\widetilde{x}'} - \widetilde{h}^2\partial_{\widetilde{x}_n}^2 + W_1(\widetilde{x}),$$

then, for any $N \geq 1$ large enough, we also have,

$$T\widetilde{P}\widetilde{v}_N(\widetilde{x}, \widetilde{\xi}; \widetilde{h}_N) = \mathcal{O}(e^{-\delta'/2\widetilde{h}_N}),$$

uniformly with respectto $h > 0$ small enough and $(\widetilde{x}, \widetilde{\xi}) \in \mathbb{R}^{2n}$ verifying $d_N(((Nk)^{1/2}\widetilde{x}', (Nk)^{2/3}\widetilde{x}_n), \mathrm{supp}\nabla\chi_N) \geq \varepsilon$.

Finally, by Proposition 7.1 and Proposition 7.3, we have the a priori estimate,

$$\|\widetilde{v}_N\|_{H^1} = \mathcal{O}(h^{-N_1}) = \mathcal{O}(e^{N_1/(N\widetilde{h})}),$$

for some $N_1 \geq 0$ independent of N, and we observe that, for $N = L/\delta_L$, one has $N_1/(\delta_L N) \to 0$ as $L \to +\infty$.

At this point, a small refinement of the propagation of the microsupport (see [FLM], Proposition 6.8) gives the existence of a constant $\delta_1 > 0$ independent of L such that, for all L large enough and $N = L/\delta_L$, one has,

$$T\widetilde{v}_N(\widetilde{x}, \widetilde{\xi}; \widetilde{h}) = \mathcal{O}(e^{-\delta_1\delta_L/\widetilde{h}}), \tag{7.12}$$

uniformly in $V(\delta_1) = \{\widetilde{x}; |\widetilde{x}| \leq \delta_1\} \times \left\{\widetilde{\xi};\ (Nk)^{\frac{1}{6}}|\widetilde{\xi}'| + |\widetilde{\xi}_n| \leq \delta_1\right\}$.

Then, using an ellipticity property of $\widetilde{p}_1$ away from $\{\widetilde{\xi};\ (Nk)^{\frac{1}{6}}|\widetilde{\xi}'| + |\widetilde{\xi}_n| \leq \delta_1\}$, and reconstructing $\widetilde{v}_N$ from $T\widetilde{v}_N$, one finally finds,

$$\|\widetilde{v}_N\|_{H^m(|\widetilde{x}| \leq \delta_2)} = \mathcal{O}(e^{-\delta_2\delta_L/\widetilde{h}}),$$

with $m \geq 0$ arbitrary, $\delta_2 > 0$ independent of L, $N = L/\delta_L$, and L arbitrarily large. Therefore, turning back to the original coordinates x and parameter h, and making $\hat{x}$ vary on all of $\partial\ddot{O}$, Theorem 7.4 follows. $\square$

8. Asymptotics of the width

As before, we denote by P_θ the distorted operator obtained from P by means of a complex distortion as in (7.2), with R_0 sufficiently large in order to have $\ddot{O} \subset \{|x| \leq R_0/2\}$. We also denote by $\mathbf{u}_\theta$ the corresponding distorted state obtain from $\mathbf{u}$ by applying the same distortion (see, e.g., [FLM] for more details).

Let $\psi_0 \in C_0^\infty([0, 2); [0, 1])$ with $\psi_0 = 1$ near $[0, 1]$, and set,

$$\psi_N(x) := \psi_0\left(\frac{\mathrm{dist}(x, \ddot{O})}{(Nk)^{2/3}}\right),$$

where, as before, $N = L/\delta_L$ with $L \geq 1$ arbitrarily large.

Then, since $\psi_N\mathbf{u} = \psi_N\mathbf{u}_\theta$, $P_\theta\mathbf{u}_\theta = \rho_1\mathbf{u}_\theta$, and $\psi_N P_\theta\psi_N\mathbf{u}_\theta = \psi_N P\psi_N\mathbf{u}$, we have,

$$\mathrm{Im}\,\rho_1\|\psi_N\mathbf{u}\|^2 = \mathrm{Im}\,\langle\psi_N P_\theta\mathbf{u}_\theta,\psi_N\mathbf{u}\rangle = \mathrm{Im}\,\langle[\psi_N,P_\theta]\mathbf{u}_\theta,\psi_N\mathbf{u}\rangle,$$

and thus,

$$\mathrm{Im}\,\rho_1 = \mathrm{Im}\,\frac{\langle 2h^2(\nabla\psi_N)\nabla\mathbf{u} + h^2(\Delta\psi_N)\mathbf{u},\psi_N\mathbf{u}\rangle + h\langle[\psi_N,R_\theta]\mathbf{u}_\theta,\psi_N\mathbf{u}\rangle}{\|\psi_N\mathbf{u}\|^2}. \tag{8.1}$$

Moreover, we know that $\|\psi_N\mathbf{u}\| = 1+\mathcal{O}(e^{-\delta/h})$ with $\delta>0$ and, by Theorem 7.4, on supp $\widetilde{\psi}_N$, we can replace $\mathbf{u}$ by $\mathbf{u}_{CN}$, up to an error $\mathcal{O}(h^Le^{-S/h})$. Also using Proposition 7.1, we deduce,

$$\begin{aligned}\mathrm{Im}\,\rho_1 = \mathrm{Im}\,&\langle 2h^2(\nabla\psi_N)\nabla\mathbf{u}_{CN} + h^2(\Delta\psi_N)\mathbf{u},\psi_N\mathbf{u}_{CN}\rangle\\ &+h\langle[\psi_N,R_\theta]\mathbf{u}_\theta,\psi_N\mathbf{u}\rangle + \mathcal{O}(h^{L-N_0})e^{-2S/h},\end{aligned} \tag{8.2}$$

for some fix $N_0\geq 0$ independent of L.

Now, we introduce $\widetilde{\psi}_0\in C_0^\infty((1,2);[0,1])$ with $\widetilde{\psi}_0=1$ near supp$\nabla\psi_0$, and we set $\widetilde{\psi}_N(x)=\widetilde{\psi}_0\left(\frac{\mathrm{dist}(x,\ddot{O})}{(Nk)^{2/3}}\right)$.

Lemma 8.1. *One has,*

$$\langle[\psi_N,R_\theta]\mathbf{u}_\theta,\psi_N\mathbf{u}\rangle = \langle\psi_N[\psi_N,R]\widetilde{\psi}_N\mathbf{u},\widetilde{\psi}_N\mathbf{u}\rangle + \mathcal{O}(h^\infty e^{-2S/h}). \tag{8.3}$$

Proof. Thanks to Assumption 3, we can make in $[\psi_N,R_\theta]$ the (complex) change of contour of integration,

$$\mathbb{R}^n\ni\xi\mapsto\xi+i\sqrt{M_0}\frac{x-y}{\sqrt{(x-y)^2+h^2}}.$$

We obtain,

$$[\psi_N,R_\theta]\mathbf{u}_\theta(x) = \frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h-\Phi/h}(\psi_N(x)-\psi_N(y))\widetilde{r}_\theta\overline{\mathbf{u}}_\theta(y)dyd\xi$$

with,

$$\Phi := \sqrt{M_0}\frac{(x-y)^2}{\sqrt{(x-y)^2+h^2}}\quad;\quad\partial_{x,y}^\alpha\partial_\xi^\beta\widetilde{r}_\theta(x,y,\xi)=\mathcal{O}(h^{-|\alpha|}\langle\xi\rangle).$$

By construction, on the set,

$$A_N := \mathrm{supp}(\psi_N(x)-\psi_N(y))\cap\{\widetilde{\psi}_N(x)\neq 1 \text{ or } \widetilde{\psi}_N(y)\neq 1\},$$

we have $|x-y|\geq c(Nk)^{2/3}$ for some constant $c>0$. As a consequence, on this set, the quantity $|x-y|/\sqrt{(x-y)^2+h^2}$ tends to 1 uniformly as $h\to 0$. Moreover, still on this set, we have either $s(x)=S$ or $s(y)=S$, and since

$|s(x)-s(y)|\leq \mu|x-y|$ with $0<\mu<\sqrt{M_0}$, we deduce the existence of a constant $c_0>0$ such that,

$$\text{For } (x,y)\in A_N,\ \text{one has } s(x)+s(y)+\Phi\geq 2S+c_0(Nk)^{2/3}.$$

Therefore, by the Calderón-Vaillancourt theorem (and also using Proposition 5.2 in order to regularize the function $s(x)$), we obtain,

$$\begin{aligned}&\|e^{-s/h}[\psi_N,R_\theta]e^{-s/h}(1-\widetilde{\psi}_N)\langle hD_n\rangle^{-1}\|\\ &\qquad+\|(1-\widetilde{\psi}_N)e^{-s/h}[\psi_N,R_\theta]e^{-s/h}\langle hD_n\rangle^{-1}\|=\mathcal{O}(h^\infty e^{-2S/h}).\end{aligned}$$

Then, writing,

$$\langle[\psi_N,R_\theta]\mathbf{u}_\theta,\psi_N\mathbf{u}\rangle=\langle e^{-s/h}[\psi_N,R_\theta]e^{-s/h}(e^{s/h}\mathbf{u}_\theta),\psi_N e^{s/h}\mathbf{u}\rangle,$$

and using Proposition 7.1 and Remark 7.2, the result follows. □

Inserting (8.3) into (8.2), and approaching $\widetilde{\psi}_N\mathbf{u}$ by $\widetilde{\psi}_N\mathbf{u}_{CN}$, we obtain,

$$\begin{aligned}\text{(8.4)}\quad \operatorname{Im}\rho_1 \;=\;& \operatorname{Im}\langle 2h^2(\nabla\psi_N)\nabla\mathbf{u}_{CN}+h^2(\Delta\psi_N)\mathbf{u},\psi_N\mathbf{u}_{CN}\rangle\\ &+h\langle\psi_N[\psi_N,R]\widetilde{\psi}_N\mathbf{u}_{CN},\widetilde{\psi}_N\mathbf{u}_{CN}\rangle+\mathcal{O}(h^{L-N_0})e^{-2S/h}.\end{aligned}$$

Finally, using Proposition 6.2 (in particular the expression (4.55) of $\mathbf{u}_{CN}$ in $\bigcup_{\gamma\in G}\mathcal{W}_N(\gamma)\cap\operatorname{supp}\widetilde{\psi}_N$), we can perform a stationary-phase expansion in (8.4) (as in [FLM], Section 7), and, for L large enough, we obtain,

$$\operatorname{Im}\rho_1=-h^{(1-n_\Gamma)/2}\sum_{j=n_0}^{L}\sum_{0\leq m\leq\ell\leq L}f_{j,\ell,m}h^{j+\ell}|\ln h|^m e^{-2S/h}+\mathcal{O}(h^{L/2})e^{-2S/h},$$

with $f_{n_0,0,0}>0$. In particular, the result for ρ_1 follows.

The result for ρ_j, $j\geq 2$ can be done along the same lines, by using a representation of $\operatorname{Im}\rho_j$ analogous to (8.1), and by approaching $\mathbf{u}$ by a linear combination of WKB expressions similar to $\mathbf{u}_{CN}$ (where the number of terms depends on the asymptotic multiplicity of the resonance: See [HeSj2], Section 10).

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